CHAPTER 6

CHAPTER 6 Polygons and Quadrilaterals

Chapter Outline

This chapter starts with the properties of polygons and narrows to focus on quadrilaterals. We will study several different types of quadrilaterals: parallelograms, rhombi, rectangles, squares, kites and trapezoids. Then, we will prove that different types of quadrilaterals are parallelograms or something more specific.

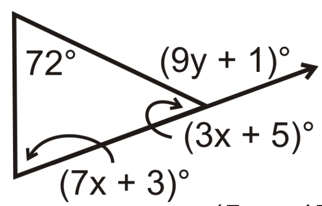
6.1 Angles in Polygons

Learning Objectives

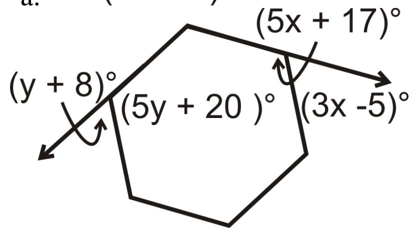
- Extend the concept of interior and exterior angles from triangles to convex polygons.
- Find the sums of interior angles in convex polygons.
- Identify the special properties of interior angles in convex quadrilaterals.

Review Queue

a. Find the measure of x and y.



a.



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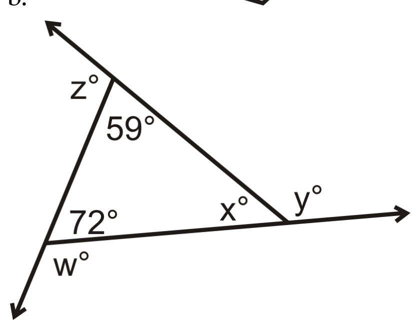


b.

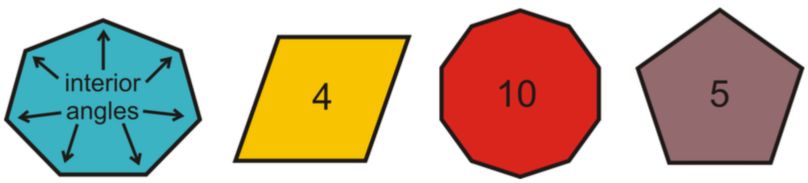
a. Find $w^\circ, x^\circ, y^\circ$, and z°.
b. What is $w^\circ + y^\circ + z^\circ$?
c. What two angles add up to y°?
d. What are $72^\circ, 59^\circ$, and x° called? What are w°, y°, and z° called?

Know What? To the right is a picture of Devil's Post pile, near Mammoth Lakes, California. These posts are cooled lava (called columnar basalt) and as the lava pools and cools, it ideally would form regular hexagonal columns. However, variations in cooling caused some columns to either not be perfect or pentagonal.

First, define regular in your own words. Then, what is the sum of the angles in a regular hexagon? What would each angle be?

Interior Angles in Convex Polygons

Recall from Chapter 4, that interior angles are the angles inside a closed figure with straight sides. Even though this concept was introduced with triangles, it can be extended to any polygon. As you can see in the images below, a polygon has the same number of interior angles as it does sides.

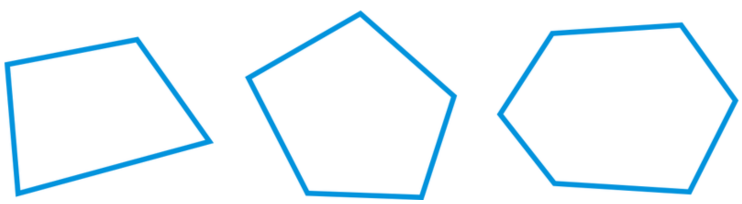


From Chapter 1, we learned that a diagonal connects two non-adjacent vertices of a convex polygon. Also, recall that the sum of the angles in a triangle is 180°. What about other polygons?

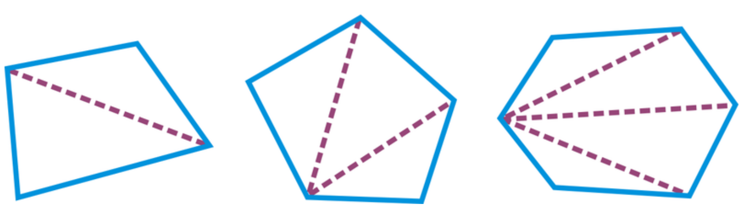
Investigation 6-1: Polygon Sum Formula

Tools Needed: paper, pencil, ruler, colored pencils (optional)

1. Draw a quadrilateral, pentagon, and hexagon.



2. Cut each polygon into triangles by drawing all the diagonals from one vertex. Count the number of triangles.



Make sure none of the triangles overlap.

3. Make a table with the information below.

Table 6.1:

Name of Polygon	***Number of Sides***	***Number of $\triangle s$ from one vertex***	***(Column 3) $\times$ ($^\circ$ in a $\triangle$)***	***Total Number of Degrees***
Quadrilateral	4	2	$2 \times 180^\circ$	360°
Pentagon	5	3	$3 \times 180^\circ$	540°
Hexagon	6	4	$4 \times 180^\circ$	720°

4. Do you see a pattern? Notice that the total number of degrees goes up by 180°. So, if the number sides is n, then the number of triangles from one vertex is $n-2$. Therefore, the formula would be $(n-2) \times 180^\circ$.

Polygon Sum Formula: For any $n-$gon, the sum of the interior angles is $(n-2) \times 180^\circ$.

Example 1: Find the sum of the interior angles of an octagon.

Solution: Use the Polygon Sum Formula and set $n = 8$.

$$(8-2) \times 180^\circ = 6 \times 180^\circ = 1080^\circ$$

Example 2: The sum of the interior angles of a polygon is 1980°. How many sides does this polygon have?

Solution: Use the Polygon Sum Formula and solve for n.

$$\begin{aligned} (n-2) \times 180^\circ &= 1980^\circ \\ 180^\circ n - 360^\circ &= 1980^\circ \\ 180^\circ n &= 2340^\circ \\ n &= 13 \qquad \text{The polygon has 13 sides.} \end{aligned}$$

Example 3: How many degrees does each angle in an equiangular nonagon have?

Solution: First we need to find the sum of the interior angles in a nonagon, set $n = 9$.

$$(9-2) \times 180^\circ = 7 \times 180^\circ = 1260^\circ$$

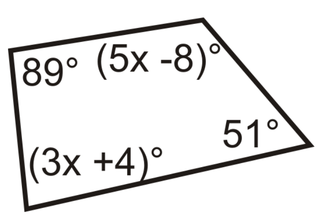
Second, because the nonagon is equiangular, every angle is equal. Dividing 1260° by 9 we get each angle is 140°.

Equiangular Polygon Formula: For any equiangular $n-$gon, the measure of each angle is $\frac{(n-2) \times 180^\circ}{n}$.

Regular Polygon: When a polygon is equilateral and equiangular.

It is important to note that in the Equiangular Polygon Formula, the word *equiangular* can be substituted with *regular*.

Example 4: ***Algebra Connection*** Find the measure of x.

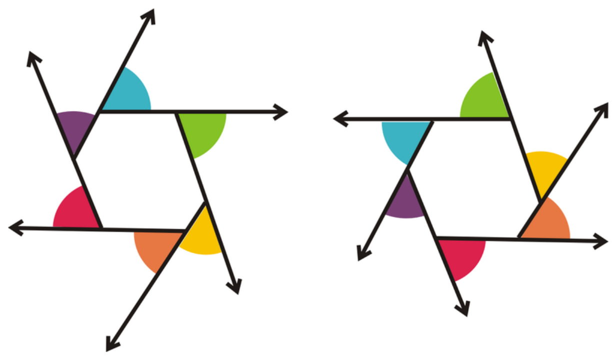


Solution: From our investigation, we found that a quadrilateral has 360°. We can write an equation to solve for x.

$$\begin{aligned}89^\circ + (5x-8)^\circ + (3x+4)^\circ + 51^\circ &= 360^\circ\\ 8x &= 224^\circ\\ x &= 28^\circ\end{aligned}$$

Exterior Angles in Convex Polygons

Recall that an exterior angle is an angle on the outside of a polygon and is formed by extending a side of the polygon (Chapter 4).



As you can see, there are two sets of exterior angles for any vertex on a polygon. It does not matter which set you use because one set is just the vertical angles of the other, making the measurement equal. In the picture to the left, the color-matched angles are vertical angles and congruent.

In Chapter 4, we introduced the Exterior Angle Sum Theorem, which stated that the exterior angles of a triangle add up to 360°. Let's extend this theorem to all polygons.

Investigation 6-2: Exterior Angle Tear-Up

Tools Needed: pencil, paper, colored pencils, scissors

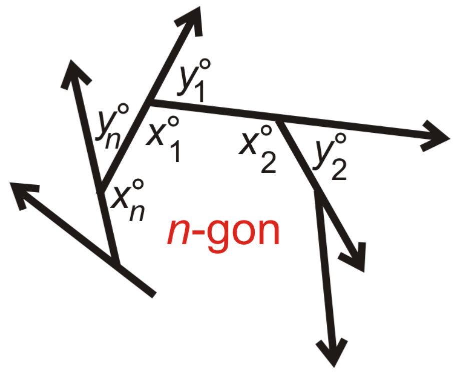
a. Draw a hexagon like the hexagons above. Color in the exterior angles as well.
b. Cut out each exterior angle and label them 1-6.

c. Fit the six angles together by putting their vertices together. What happens?

The angles all fit around a point, meaning that the exterior angles of a hexagon add up to 360°, just like a triangle. We can say this is true for all polygons.

Exterior Angle Sum Theorem: The sum of the exterior angles of any polygon is 360°.

Proof of the Exterior Angle Sum Theorem



Given: Any $n-$gon with n sides, n interior angles and n exterior angles.

Prove: n exterior angles add up to 360°

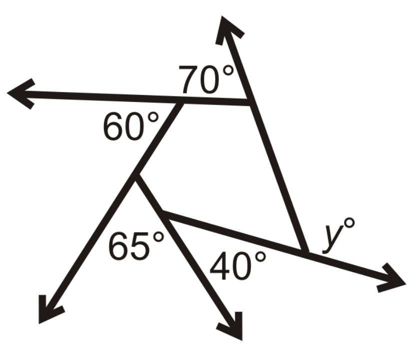
NOTE: The interior angles are $x_1, x_2, \ldots x_n$.

The exterior angles are $y_1, y_2, \ldots y_n$.

TABLE 6.2:

Statement	***Reason***
1. Any $n-$gon with n sides, n interior angles and n exterior angles.	Given
2. x_n° *and* y_n° are a linear pair	Definition of a linear pair
3. x_n° *and* y_n° are supplementary	Linear Pair Postulate
4. $x_n^\circ + y_n^\circ = 180^\circ$	Definition of supplementary angles
5. $(x_1^\circ + x_2^\circ + \ldots + x_n^\circ) + (y_1^\circ + y_2^\circ + \ldots + y_n^\circ) = 180^\circ n$	Sum of all interior and exterior angles in an $n-$gon
6. $(n-2)180^\circ = (x_1^\circ + x_2^\circ + \ldots + x_n^\circ)$	Polygon Sum Formula
7. $180^\circ n = (n-2)180^\circ + (y_1^\circ + y_2^\circ + \ldots + y_n^\circ)$	Substitution PoE
8. $180^\circ n = 180^\circ n - 360^\circ + (y_1^\circ + y_2^\circ + \ldots + y_n^\circ)$	Distributive PoE
9. $360^\circ = (y_1^\circ + y_2^\circ + \ldots + y_n^\circ)$	Subtraction PoE

Example 5: What is y?



Solution: y is an exterior angle, as well as all the other given angle measures. Exterior angles add up to 360°, so set up an equation.

$$70^\circ + 60^\circ + 65^\circ + 40^\circ + y = 360^\circ$$
$$y = 125^\circ$$

Example 6: What is the measure of each exterior angle of a regular heptagon?

Solution: Because the polygon is regular, each interior angle is equal. This also means that all the exterior angles are equal. The exterior angles add up to 360°, so each angle is $\frac{360^\circ}{7} \approx 51.43^\circ$.

Know What? Revisited A regular polygon has congruent sides and angles. A regular hexagon has $(6-2)180^\circ = 4 \cdot 180^\circ = 720^\circ$ total degrees. Each angle would be 720° divided by 6 or 120°.

Review Questions

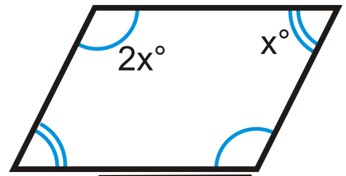
1. Fill in the table.

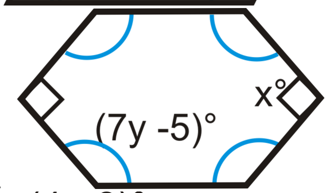
TABLE 6.3:

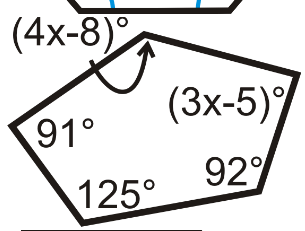
# of sides	# of $\triangle s$ from one vertex	$\triangle s \times 180^\circ$ (sum)	Each angle in a *regular* $n-$gon	Sum of the *exterior* angles
3	1	180°	60°	
4	2	360°	90°	
5	3	540°	108°	
6	4	720°	120°	
7				
8				
9				
10				
11				
12				

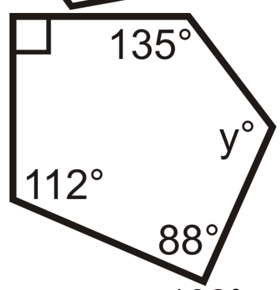
2. What is the sum of the angles in a 15-gon?
3. What is the sum of the angles in a 23-gon?
4. The sum of the interior angles of a polygon is 4320°. How many sides does the polygon have?
5. The sum of the interior angles of a polygon is 3240°. How many sides does the polygon have?
6. What is the measure of each angle in a regular 16-gon?
7. What is the measure of each angle in an equiangular 24-gon?
8. What is the measure of each exterior angle of a dodecagon?
9. What is the measure of each exterior angle of a 36-gon?
10. What is the sum of the exterior angles of a 27-gon?
11. If the measure of one interior angle of a regular polygon is 160°, how many sides does it have?
12. How many sides does a regular polygon have if the measure of one of its interior angles is 168°?
13. If the measure of one interior angle of a regular polygon is $158\frac{14}{17}^\circ$, how many sides does it have?
14. How many sides does a regular polygon have if the measure of one exterior angle is 15°?
15. If the measure of one exterior angle of a regular polygon is 36°, how many sides does it have?
16. How many sides does a regular polygon have if the measure of one exterior angle is $32\frac{8}{11}^\circ$?

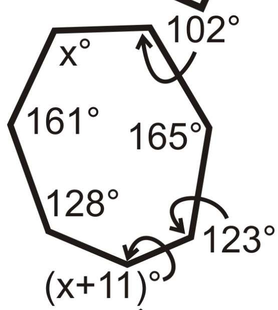
For questions 11-20, find the measure of the missing variable(s).

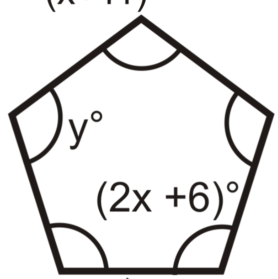
17. 

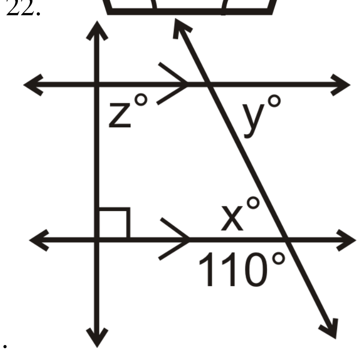
18. 

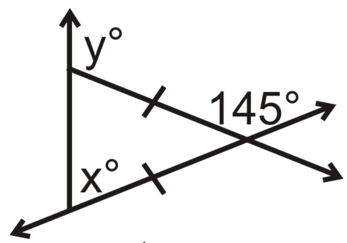
19. 

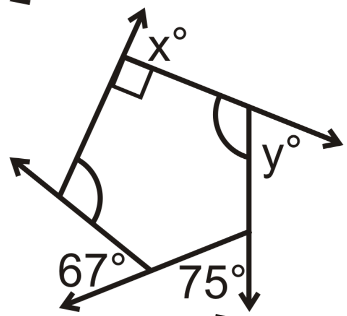
20. 

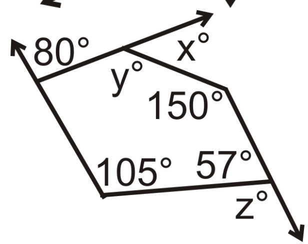
21. 

22. 

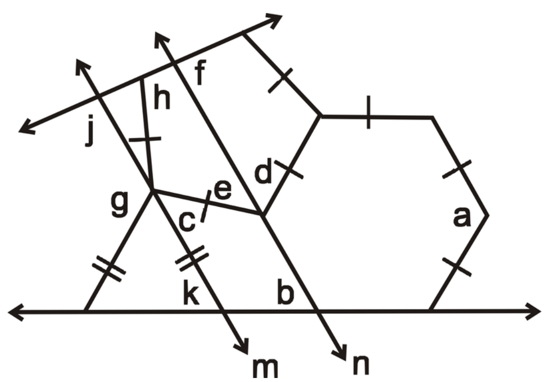
23. 

24. 

25. 

26. 

27. The interior angles of a pentagon are $x^\circ, x^\circ, 2x^\circ, 2x^\circ$, and $2x^\circ$. What is the measure of the larger angles?
28. The exterior angles of a quadrilateral are $x^\circ, 2x^\circ, 3x^\circ$, and $4x^\circ$. What is the measure of the smallest angle?
29. The interior angles of a hexagon are $x^\circ, (x+1)^\circ, (x+2)^\circ, (x+3)^\circ, (x+4)^\circ$, and $(x+5)^\circ$. What is x?
30. ***Challenge*** Each interior angle forms a linear pair with an exterior angle. In a regular polygon you can use two different formulas to find the measure of each exterior angle. One way is $\frac{360^\circ}{n}$ and the other is $180^\circ - \frac{(n-2)180^\circ}{n}$ (180° minus Equiangular Polygon Formula). Use algebra to show these two expressions are equivalent.
31. ***Angle Puzzle*** Find the measures of the lettered angles below given that $m \parallel n$.



Review Queue Answers

a. $$\begin{aligned} 72^\circ + (7x+3)^\circ + (3x+5)^\circ &= 180^\circ \\ 10x + 80^\circ &= 180^\circ \\ 10x &= 100^\circ \\ x &= 10^\circ \end{aligned}$$

b. $$\begin{aligned} (5x+17)^\circ + (3x-5)^\circ &= 180^\circ \\ 8x + 12^\circ &= 180^\circ \end{aligned}$$

$$8x = 168^\circ$$
$$x = 21^\circ$$

a. $w = 108^\circ, x = 49^\circ, y = 131^\circ, z = 121^\circ$
b. 360°
c. $59^\circ + 72^\circ$
d. interior angles, exterior angles

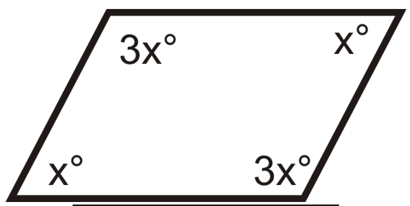
6.2 Properties of Parallelograms

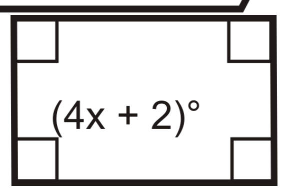
Learning Objectives

- Define a parallelogram.
- Understand the properties of a parallelogram
- Apply theorems about a parallelogram's sides, angles and diagonals.

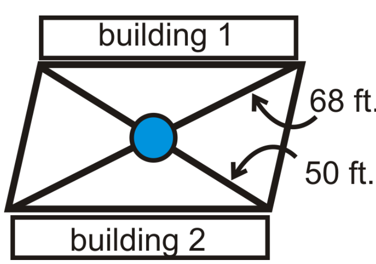
Review Queue

a. Draw a quadrilateral with one set of parallel sides.
b. Draw a quadrilateral with two sets of parallel sides.
c. Find the measure of the missing angles in the quadrilaterals below.

a. 

b. 

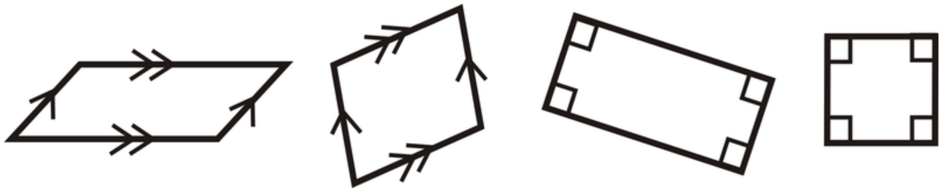
Know What? A college has a parallelogram-shaped courtyard between two buildings. The school wants to build two walkways on the diagonals of the parallelogram with a fountain where they intersect. The walkways are going to be 50 feet and 68 feet long. Where would the fountain be?



What is a Parallelogram?

Parallelogram: A quadrilateral with two pairs of parallel sides.

Here are some examples:

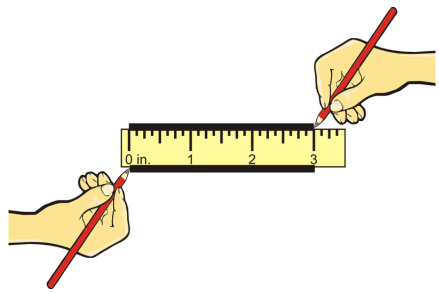


Notice that each pair of sides is marked parallel. As is the case with the rectangle and square, recall that two lines are parallel when they are perpendicular to the same line. Once we know that a quadrilateral is a parallelogram, we can discover some additional properties.

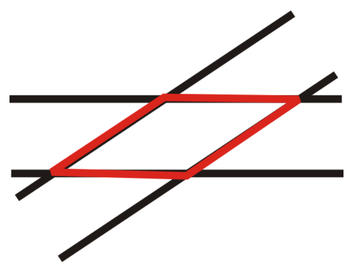
Investigation 6-2: Properties of Parallelograms

Tools Needed: Paper, pencil, ruler, protractor

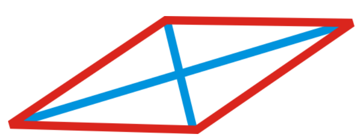
a. Draw a set of parallel lines by placing your ruler on the paper and drawing a line on either side of it. Make your lines 3 inches long.



b. Rotate the ruler and repeat this so that you have a parallelogram. Your second set of parallel lines can be any length. If you have colored pencils, outline the parallelogram in another color.



c. Measure the four interior angles of the parallelogram as well as the length of each side. Can you conclude anything about parallelograms, other than opposite sides are parallel?
d. Draw the diagonals. Measure each and then measure the lengths from the point of intersection to each vertex.



To continue to explore the properties of a parallelogram, see the website:

http://www.mathwarehouse.com/geometry/quadrilaterals/parallelograms/interactive-parallelogram.php

In the above investigation, we drew a parallelogram. From this investigation we can conclude:

- The sides that are parallel are also congruent.
- Opposite angles are congruent.
- Consecutive angles are supplementary.

- The diagonals bisect each other.

Opposite Sides Theorem: If a quadrilateral is a parallelogram, then the opposite sides are congruent.

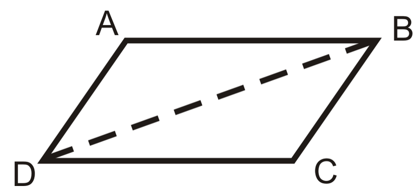
Opposite Angles Theorem: If a quadrilateral is a parallelogram, then the opposite angles are congruent.

Consecutive Angles Theorem: If a quadrilateral is a parallelogram, then the consecutive angles are supplementary.

Parallelogram Diagonals Theorem: If a quadrilateral is a parallelogram, then the diagonals bisect each other.

To prove the first three theorems, one of the diagonals must be added to the figure and then the two triangles can be proved congruent.

Proof of Opposite Sides Theorem



Given: $ABCD$ is a parallelogram with diagonal $\overline{BD}$

Prove: $\overline{AB} \cong \overline{DC}$, $\overline{AD} \cong \overline{BC}$

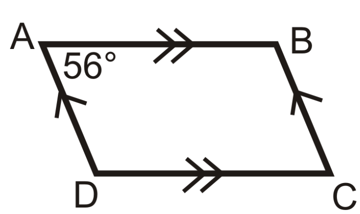
TABLE 6.4:

Statement	***Reason***
1. $ABCD$ is a parallelogram with diagonal $\overline{BD}$	Given
2. $\overline{AB} \parallel \overline{DC}$, $\overline{AD} \parallel \overline{BC}$	Definition of a parallelogram
3. $\angle ABD \cong BDC$, $\angle ADB \cong DBC$	Alternate Interior Angles Theorem
4. $\overline{DB} \cong \overline{DB}$	Reflexive PoC
5. $\triangle ABD \cong \triangle CDB$	ASA
6. $\overline{AB} \cong \overline{DC}$, $\overline{AD} \cong \overline{BC}$	CPCTC

The proof of the Opposite Angles Theorem is almost identical. For the last step, the angles are congruent by CPCTC. You will prove the other three theorems in the review questions.

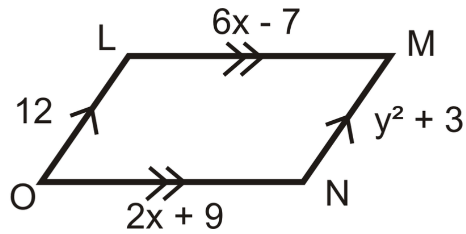
Example 1: $ABCD$ is a parallelogram. If $m\angle A = 56^\circ$, find the measure of the other three angles.

Solution: Draw a picture. When labeling the vertices, the letters are listed, in order, clockwise.



If $m\angle A = 56^\circ$, then $m\angle C = 56^\circ$ because they are opposite angles. $\angle B$ and $\angle D$ are consecutive angles with $\angle A$, so they are both supplementary to $\angle A$. $m\angle A + m\angle B = 180^\circ, 56^\circ + m\angle B = 180^\circ, m\angle B = 124^\circ$. $m\angle D = 124^\circ$.

Example 2: ***Algebra Connection*** Find the values of x and y.



Solution: Opposite sides are congruent, so we can set each pair equal to each other and solve both equations.

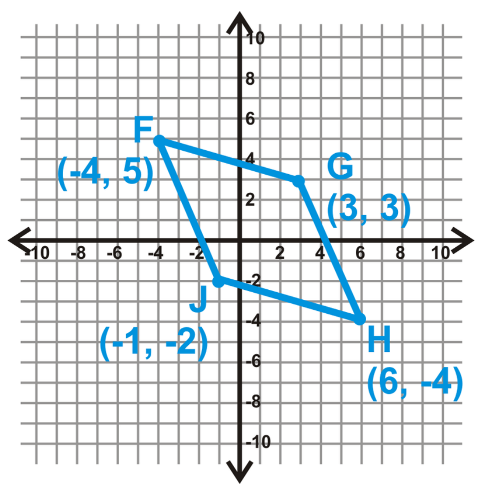
$$\begin{aligned} 6x-7 &= 2x+9 \\ 4x &= 16 \\ x &= 4 \end{aligned} \qquad\qquad \begin{aligned} y^2+3 &= 12 \\ y^2 &= 9 \\ y &= 3 \ or \ -3 \end{aligned}$$

Even though $y = 3$ or -3, lengths cannot be negative, so $y = 3$.

Diagonals in a Parallelogram

From the Parallelogram Diagonals Theorem, we know that the diagonals of a parallelogram bisect each other.

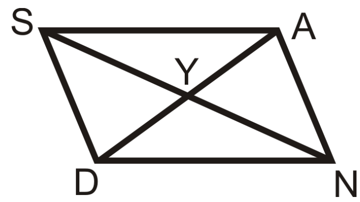
Example 3: Show that the diagonals of $FGHJ$ bisect each other.



Solution: The easiest way to show this is to find the midpoint of each diagonal. If it is the same point, you know they intersect at each other's midpoint and, by definition, cuts a line in half.

$$\text{Midpoint of } \overline{FH} : \left(\frac{-4+6}{2}, \frac{5-4}{2}\right) = (1, 0.5)$$
$$\text{Midpoint of } \overline{GJ} : \left(\frac{3-1}{2}, \frac{3-2}{2}\right) = (1, 0.5)$$

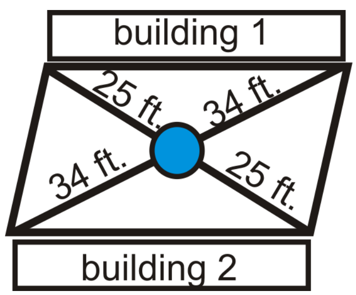
Example 4: ***Algebra Connection*** $SAND$ is a parallelogram and $SY = 4x - 11$ and $YN = x + 10$. Solve for x.



Solution: $\overline{AD}$ and $\overline{SN}$ bisect each other, so $SY = YN$.

$$\begin{aligned} 4x - 11 &= x + 10 \\ 3x &= 21 \\ x &= 7 \end{aligned}$$

Know What? Revisited By the Parallelogram Diagonals Theorem, the fountain is going to be 34 feet from either endpoint on the 68 foot diagonal and 25 feet from either endpoint on the 50 foot diagonal.

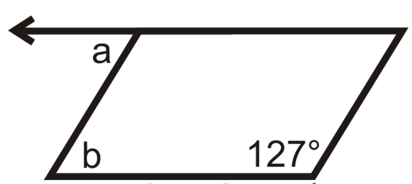


Review Questions

1. If $m\angle B = 72^\circ$ in parallelogram $ABCD$, find the other three angles.
2. If $m\angle S = 143^\circ$ in parallelogram $PQRS$, find the other three angles.
3. If $\overline{AB} \perp \overline{BC}$ in parallelogram $ABCD$, find the measure of all four angles.
4. If $m\angle F = x^\circ$ in parallelogram $EFGH$, find expressions for the other three angles in terms of x.

For questions 5-13, find the measures of the variable(s). All the figures below are parallelograms.

5.



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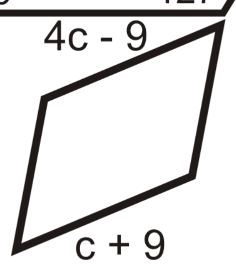
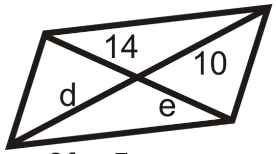
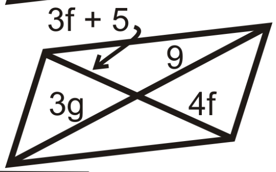
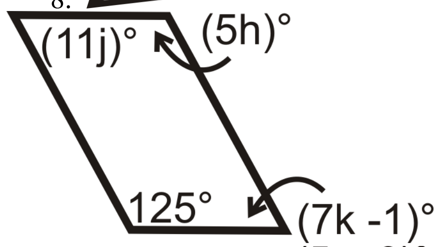
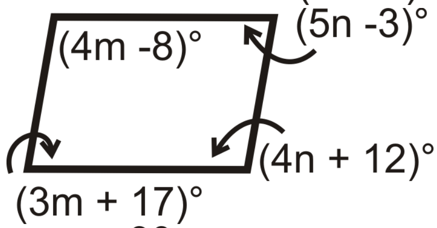
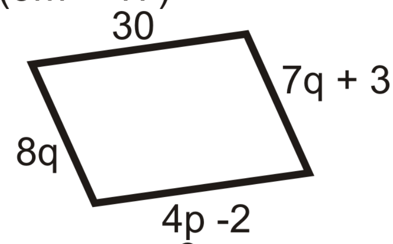


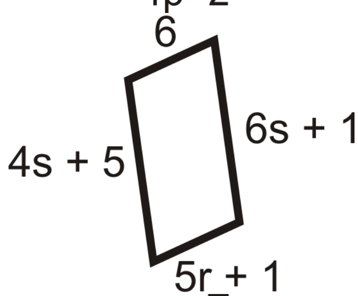
7. 

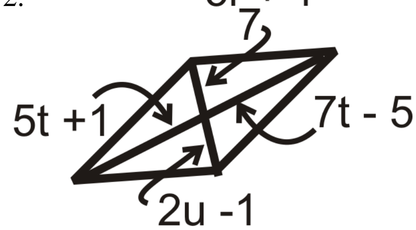
8. 

9. 

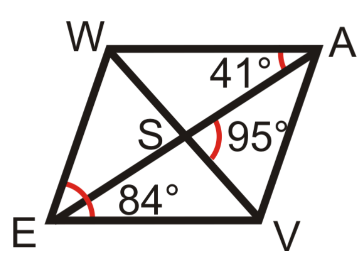
10. 

11. 

12. 

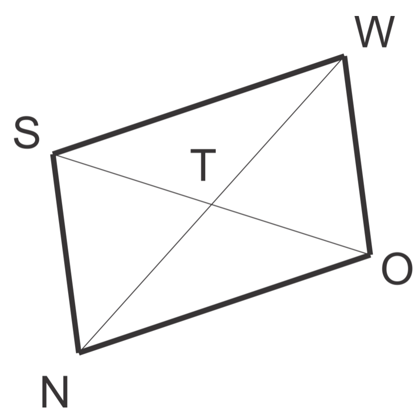
13. 

Use the parallelogram $WAVE$ to find:



14. $m\angle AWE$
15. $m\angle ESV$
16. $m\angle WEA$
17. $m\angle AVW$

In the parallelogram $SNOW, ST = 6, NW = 4, m\angle OSW = 36^\circ, m\angle SNW = 58^\circ$ and $m\angle NTS = 80^\circ$. (*diagram is not drawn to scale*)



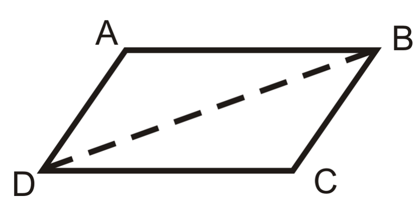
18. SO
19. NT
20. $m\angle NWS$
21. $m\angle SOW$

Plot the points $E(-1,3), F(3,4), G(5,-1), H(1,-2)$ and use parallelogram $EFGH$ for problems 22-25.

22. Find the coordinates of the point at which the diagonals intersect. How did you do this?
23. Find the slopes of all four sides. What do you notice?
24. Use the distance formula to find the lengths of all four sides. What do you notice?
25. Make a conjecture about how you might determine whether a quadrilateral in the coordinate is a parallelogram.

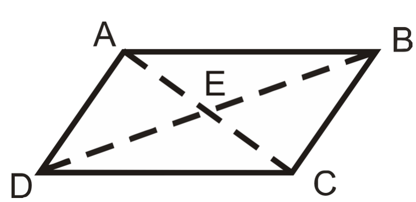
Write a two-column proof.

26. ***Opposite Angles Theorem***



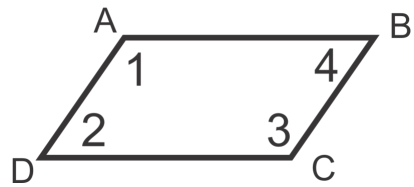
Given: $ABCD$ is a parallelogram with diagonal $\overline{BD}$Prove: $\angle A \cong \angle C$

27. ***Parallelogram Diagonals Theorem***



Given: $ABCD$ is a parallelogram with diagonals $\overline{BD}$ and $\overline{AC}$Prove: $\overline{AE} \cong \overline{EC}$, $\overline{DE} \cong \overline{EB}$

28. Fill in the blanks for the proof of the ***Consecutive Angles Theorem***

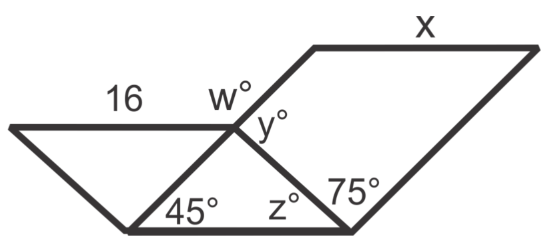


Given: $ABCD$ is a parallelogram Prove: $m\angle 1 + m\angle 2 = 180^\circ$

TABLE 6.5:

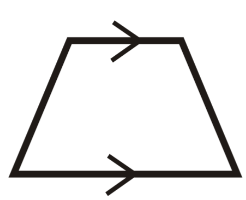
Statements	***Reasons***
1.	Given
2. $m\angle 1 = m\angle 3$ and ________	
3. $m\angle 1 + m\angle 2 + m\angle 3 + m\angle 4 = 360^\circ$	
4. $m\angle 1 + m\angle 2 + m\angle 1 + m\angle 2 = 360^\circ$	
5. $2(m\angle 1 + m\angle 2) = 360^\circ$	
6.	Division POE

Use the diagram below to find the indicated lengths or angle measures for problems 29-32. The two quadrilaterals that share a side are parallelograms.

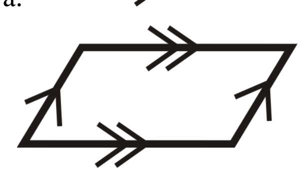


29. w
30. x
31. y
32. z

Review Queue Answers



a.



b.

c. $$\begin{aligned} 3x+x+3x+x &= 360^\circ \\ 8x &= 360^\circ \\ x &= 45^\circ \end{aligned}$$

d. $$\begin{aligned} 4x+2 &= 90^\circ \\ 4x &= 88^\circ \\ x &= 22^\circ \end{aligned}$$

6.3 Proving Quadrilaterals are Parallelograms

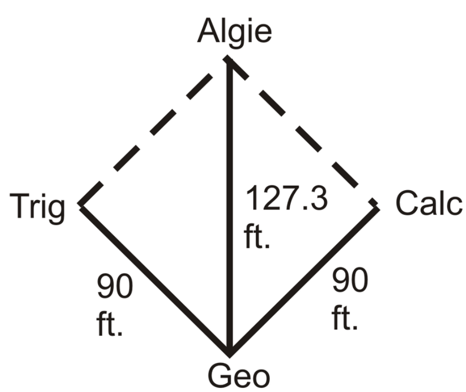
Learning Objectives

- Prove a quadrilateral is a parallelogram using the converses of the theorems from the previous section.
- Prove a quadrilateral is a parallelogram in the coordinate plane.

Review Queue

a. Write the converses of: the Opposite Sides Theorem, Opposite Angles Theorem, Consecutive Angles Theorem and the Parallelogram Diagonals Theorem.
b. Are any of these converses true? If not, find a counterexample.
c. Plot the points $A(2,2)$, $B(4,-2)$, $C(-2,-4)$, and $D(-6,-2)$.
   a. Find the slopes of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$. Is $ABCD$ a parallelogram?
   b. Find the point of intersection of the diagonals. Does this go along with what you found in part a?

Know What? Four friends, Geo, Trig, Algie, and Calc are marking out a baseball diamond. Geo is standing at home plate. Trig is 90 feet away at 3^{rd} base, Algie is 127.3 feet away at 2^{nd} base, and Calc is 90 feet away at 1^{st} base. The angle at home plate is 90°, from 1^{st} to 3^{rd} is 90°. Find the length of the other diagonal and determine if the baseball diamond is a parallelogram. If it is, what kind of parallelogram is it?



Determining if a Quadrilateral is a Parallelogram

In the last section, we introduced the Opposite Sides Theorem, Opposite Angles Theorem, Consecutive Angles Theorem and the Parallelogram Diagonals Theorem. #1 in the Review Queue above, had you write the converses of each of these:

Opposite Sides Theorem Converse: If the opposite sides of a quadrilateral are congruent, then the figure is a parallelogram.

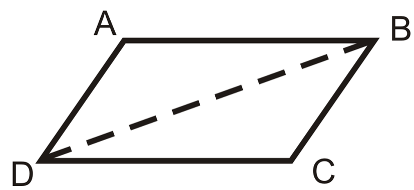
Opposite Angles Theorem Converse: If the opposite angles of a quadrilateral are congruent, then the figure is a parallelogram.

Consecutive Angles Theorem Converse: If the consecutive angles of a quadrilateral are supplementary, then the figure is a parallelogram.

Parallelogram Diagonals Theorem Converse: If the diagonals of a quadrilateral bisect each other, then the figure is a parallelogram.

Are these converses true? The answer is yes. Each of these converses can be a way to show that a quadrilateral is a parallelogram. However, the Consecutive Angles Converse can be a bit tricky, considering you would have to show that each angle is supplementary to its neighbor ($\angle A$ and $\angle B$, $\angle B$ and $\angle C$, $\angle C$ and $\angle D$, and $\angle A$ and $\angle D$). We will not use this converse.

Proof of the Opposite Sides Theorem Converse



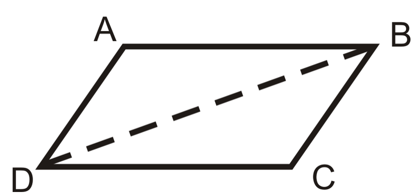
Given: $\overline{AB} \cong \overline{DC}$, $\overline{AD} \cong \overline{BC}$

Prove: $ABCD$ is a parallelogram

TABLE 6.6:

Statement	***Reason***
1. $\overline{AB} \cong \overline{DC}$, $\overline{AD} \cong \overline{BC}$	Given
2. $\overline{DB} \cong \overline{DB}$	Reflexive PoC
3. $\triangle ABD \cong \triangle CDB$	SSS
4. $\angle ABD \cong \angle BDC$, $\angle ADB \cong \angle DBC$	CPCTC
5. $\overline{AB} \mid\mid \overline{DC}$, $\overline{AD} \mid\mid \overline{BC}$	Alternate Interior Angles Converse
6. $ABCD$ is a parallelogram	Definition of a parallelogram

Example 1: Write a two-column proof.



Given: $\overline{AB} \mid\mid \overline{DC}$ and $\overline{AB} \cong \overline{DC}$

Prove: $ABCD$ is a parallelogram

Solution:

TABLE 6.7:

Statement	***Reason***
1. $\overline{AB} \mid\mid \overline{DC}$ and $\overline{AB} \cong \overline{DC}$	Given
2. $\angle ABD \cong \angle BDC$	Alternate Interior Angles

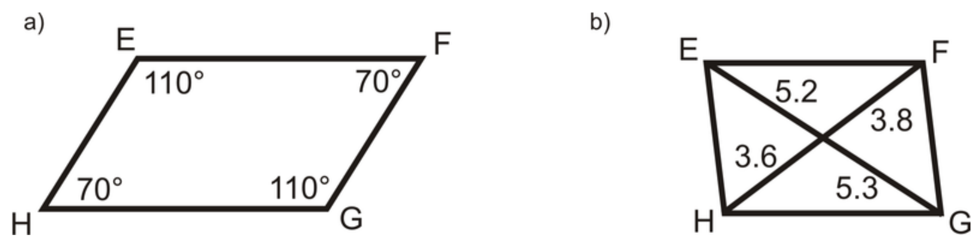
TABLE 6.7: (continued)

Statement	***Reason***
3. $\overline{DB} \cong \overline{DB}$	Reflexive PoC
4. $\triangle ABD \cong \triangle CDB$	SAS
5. $\overline{AD} \cong \overline{BC}$	CPCTC
6. $ABCD$ is a parallelogram	Opposite Sides Converse

Example 1 proves an additional way to show that a quadrilateral is a parallelogram.

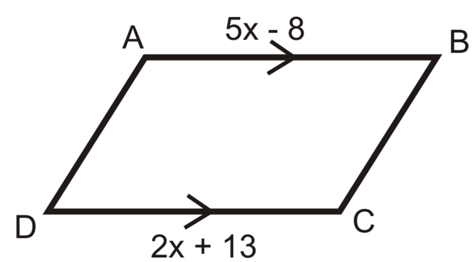
Theorem 5-10: If a quadrilateral has one set of parallel lines that are also congruent, then it is a parallelogram.

Example 2: Is quadrilateral $EFGH$ a parallelogram? How do you know?



Solution: For part a, the opposite angles are equal, so by the Opposite Angles Theorem Converse, $EFGH$ is a parallelogram. In part b, the diagonals do not bisect each other, so $EFGH$ is not a parallelogram.

Example 3: ***Algebra Connection*** What value of x would make $ABCD$ a parallelogram?



Solution: $\overline{AB} \,||\, \overline{DC}$ from the markings. By Theorem 5-10, $ABCD$ would be a parallelogram if $AB = DC$ as well.

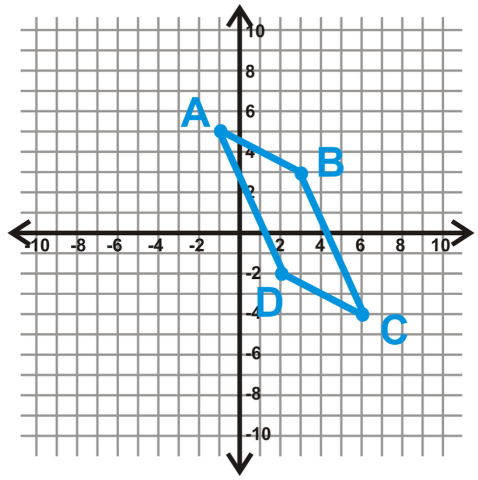
$$\begin{aligned} 5x - 8 &= 2x + 13 \\ 3x &= 21 \\ x &= 7 \end{aligned}$$

In order for $ABCD$ to be a parallelogram, x must equal 7.

Showing a Quadrilateral is a Parallelogram in the Coordinate Plane

To show that a quadrilateral is a parallelogram in the $x - y$ plane, you will need to use a combination of the slope formulas, the distance formula and the midpoint formula. For example, to use the Definition of a Parallelogram, you would need to ***find the slope of all four sides*** to see if the opposite sides are parallel. To use the Opposite Sides Converse, you would have to find the length (***using the distance formula***) of each side to see if the opposite sides are congruent. To use the Parallelogram Diagonals Converse, you would need to use the ***midpoint formula*** for each diagonal to see if the midpoint is the same for both. Finally, you can use Theorem 5-10 in the coordinate plane. To use this theorem, you would need to show that one pair of opposite sides has the same slope (***slope formula***) and the same length (***distance formula***).

Example 4: Is the quadrilateral $ABCD$ a parallelogram?



Solution: We have determined there are four different ways to show a quadrilateral is a parallelogram in the $x-y$ plane. Let's use Theorem 5-10. First, find the length of AB and CD.

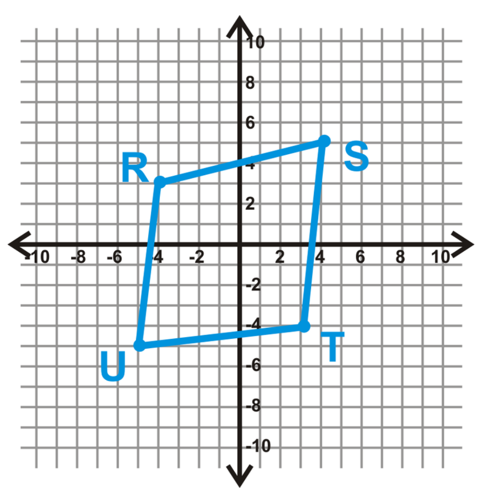
$$\begin{aligned} AB &= \sqrt{(-1-3)^2+(5-3)^2} \\ &= \sqrt{(-4)^2+2^2} \\ &= \sqrt{16+4} \\ &= \sqrt{20} \end{aligned} \qquad \begin{aligned} CD &= \sqrt{(2-6)^2+(-2+4)^2} \\ &= \sqrt{(-4)^2+2^2} \\ &= \sqrt{16+4} \\ &= \sqrt{20} \end{aligned}$$

$AB = CD$, so if the two lines have the same slope, $ABCD$ is a parallelogram.

Slope $AB = \frac{5-3}{-1-3} = \frac{2}{-4} = -\frac{1}{2}$ Slope $CD = \frac{-2+4}{2-6} = \frac{2}{-4} = -\frac{1}{2}$

By Theorem 5-10, $ABCD$ is a parallelogram.

Example 5: Is the quadrilateral $RSTU$ a parallelogram?



Solution: Let's use the Parallelogram Diagonals Converse to determine if $RSTU$ is a parallelogram. Find the midpoint of each diagonal.

Midpoint of $RT = \left(\frac{-4+3}{2}, \frac{3-4}{2}\right) = (-0.5, -0.5)$

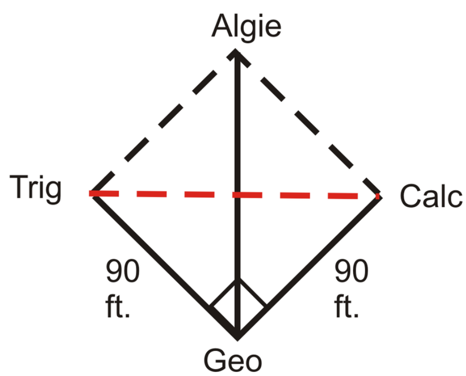
Midpoint of $SU = \left(\frac{4-5}{2}, \frac{5-5}{2}\right) = (-0.5, 0)$

Because the midpoint is not the same, $RSTU$ is not a parallelogram.

Know What? Revisited First, we can use the Pythagorean Theorem to find the length of the second diagonal.

$$\begin{aligned} 90^2 + 90^2 &= d^2 \\ 8100 + 8100 &= d^2 \\ 16200 &= d^2 \\ d &= 127.3 \end{aligned}$$

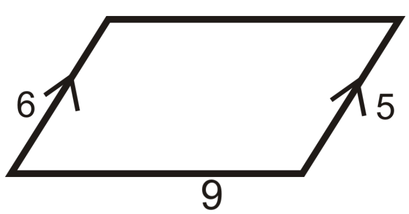
This means that the diagonals are equal. If the diagonals are equal, the other two sides of the diamond are also 90 feet. Therefore, the baseball diamond is a parallelogram, and more specifically, it is a square.



Review Questions

For questions 1-12, determine if the quadrilaterals are parallelograms. If they are, write a reason.

1.



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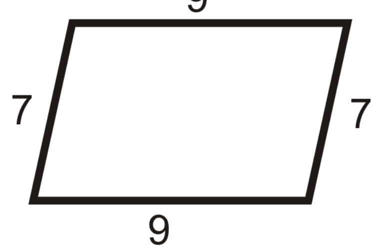
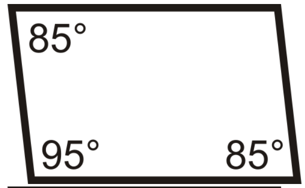
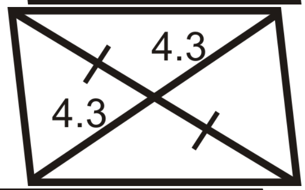
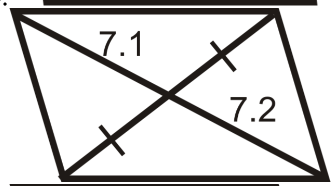


3. 

4. 

5. 

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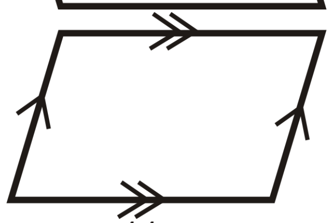
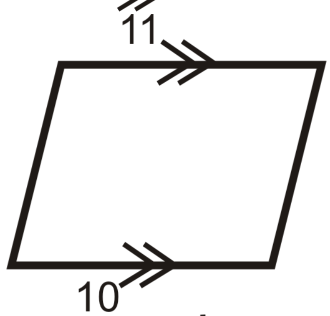
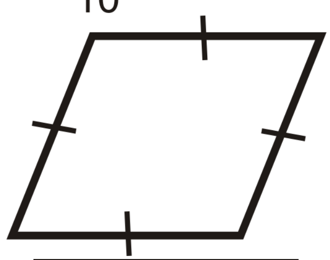
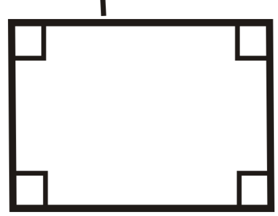
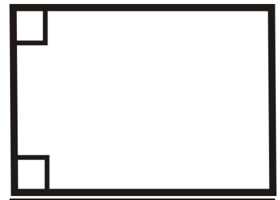
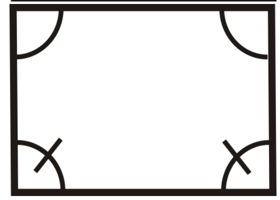
7. 

8. 

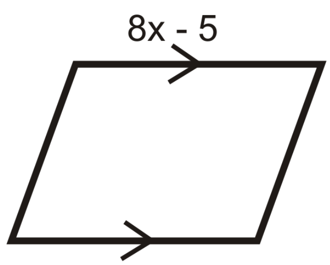
9. 

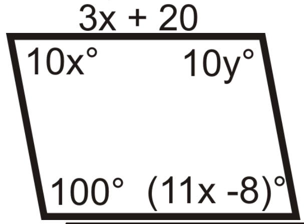
10. 

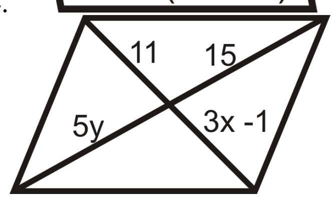
11. 

12. 

For questions 13-15, determine the value of x and y that would make the quadrilateral a parallelogram.

13. 

14. 

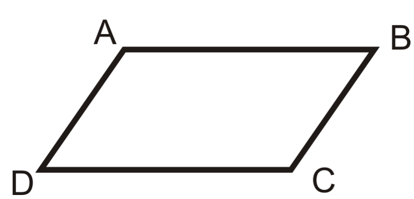
15. 

For questions 16-18, determine if $ABCD$ is a parallelogram.

16. $A(8,-1),\ B(6,5),\ C(-7,2),\ D(-5,-4)$
17. $A(-5,8),\ B(-2,9),\ C(3,4),\ D(0,3)$
18. $A(-2,6),\ B(4,-4),\ C(13,-7),\ D(4,-10)$

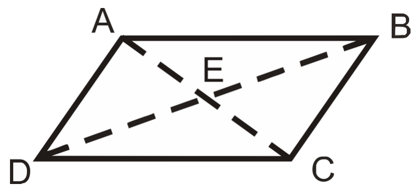
Write a two-column proof.

19. ***Opposite Angles Theorem Converse***

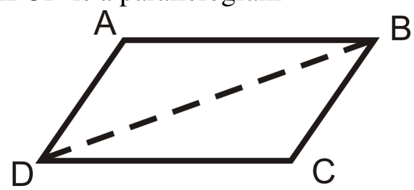


Given: $\angle A \cong \angle C,\ \angle D \cong \angle B$Prove: $ABCD$ is a parallelogram

20. ***Parallelogram Diagonals Theorem Converse***



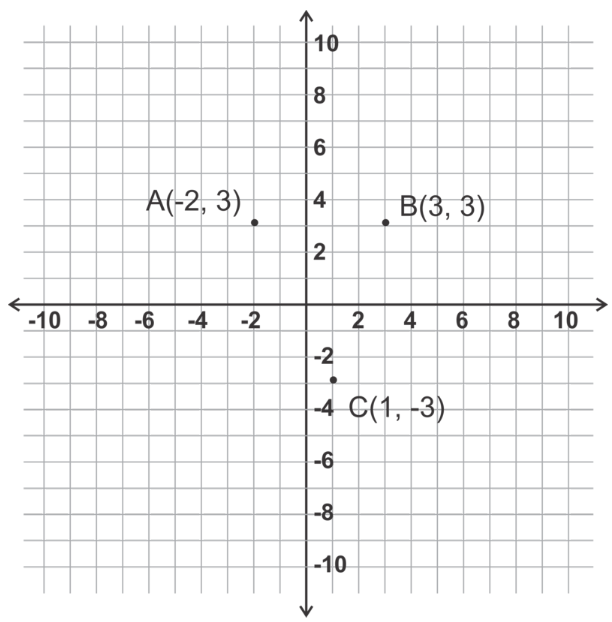
Given: $\overline{AE} \cong \overline{EC}$, $\overline{DE} \cong \overline{EB}$Prove: $ABCD$ is a parallelogram



21\.

Given: $\angle ADB \cong CBD$, $\overline{AD} \cong \overline{BC}$Prove: $ABCD$ is a parallelogram

Suppose that $A(-2,3)$, $B(3,3)$ and $C(1,-3)$ are three of four vertices of a parallelogram.



22. Depending on where you choose to put point D, the name of the parallelogram you draw will change. Sketch a picture to show all possible parallelograms. How many can you draw?
23. If you know the parallelogram is named $ABDC$, what is the slope of side parallel to $\overline{AC}$?
24. Again, assuming the parallelogram is named $ABDC$, what is the length of $\overline{BD}$?
25. Find the points of intersection of the diagonals of the three parallelograms formed. Label them X in parallelogram $ABCD$, Y in parallelogram $ADBC$ and Z in parallelogram $ABDC$.
26. Connect the points X, Y and Z to form a triangle. What do you notice about this triangle?

The points $Q(-1,1)$, $U(7,1)$, $A(1,7)$ and $D(-1,5)$ are the vertices of quadrilateral $QUAD$. Plot the points on graph paper to complete problems 27-30.

27. Find the midpoints of sides $\overline{QU}$, $\overline{UA}$, $\overline{AD}$ and $\overline{DQ}$. Label them W, X, Y and Z respectively.
28. Connect the midpoints to form quadrilateral $WXYZ$. What does this quadrilateral appear to be?
29. Use slopes to verify your answer to problem 29.
30. Use midpoints to verify your answer to problem 29.
31. This phenomenon occurs in all quadrilaterals. Describe how you might prove this fact. (Hint: each side of quadrilateral $WXYZ$ is a midsegment in a triangle formed by two sides of the parallelogram and a diagonal.)

Review Queue Answers

1. **Opposite Sides Theorem Converse:** If the opposite sides of a quadrilateral are congruent, then the figure is a parallelogram.

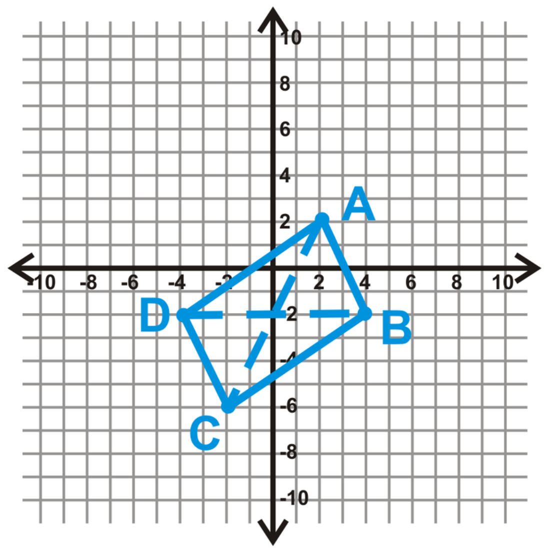
Opposite Angles Theorem Converse: If the opposite angles of a quadrilateral are congruent, then the figure is a parallelogram.

Consecutive Angles Theorem Converse: If the consecutive angles of a quadrilateral are supplementary, then the figure is a parallelogram.

Parallelogram Diagonals Theorem Converse: If the diagonals of a quadrilateral bisect each other, then the figure is a parallelogram.

2. All the converses are true.

3.



a) Slope $AB =$ Slope $CD = -\frac{1}{2}$
Slope $AD =$ Slope $BC = \frac{2}{3}$

ABCD is a parallelogram because the opposite sides are parallel.

b) Midpoint of $BD = (0, -2)$
Midpoint of $AC = (0, -2)$

Yes, the midpoint of the diagonals are the same, so they bisect each other. This corresponds with what we found in part a.

6.4 Rectangles, Rhombuses and Squares

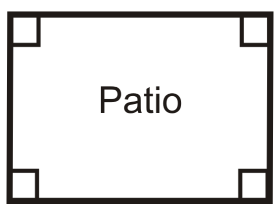
Learning Objectives

- Define and analyze a rectangle, rhombus, and square.
- Determine if a parallelogram is a rectangle, rhombus, or square in the coordinate plane.
- Analyze the properties of the diagonals of a rectangle, rhombus, and square.

Review Queue

a. Define rectangle in your own words. Is a rectangle a parallelogram?
b. Define square in your own words. Is a square a parallelogram? Is it a rectangle?
c. List five examples where you might see a square, rectangle, or rhombus in real life.

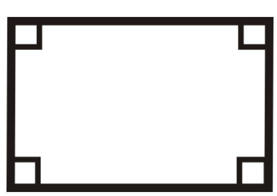
Know What? You are designing a patio for you backyard. You decide to mark it off using your tape measure. Two sides are ***21 feet*** long and two sides are ***28 feet*** long. Explain how you would only use the tape measure to make your patio a rectangle.



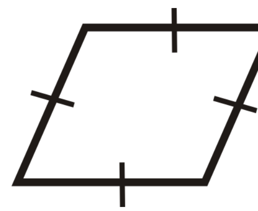
Defining Special Parallelograms

Rectangles, Rhombuses (the plural is also Rhombi) and Squares are all more specific versions of parallelograms, also called special parallelograms. Taking the theorems we learned in the previous two sections, we have three more new theorems.

Rectangle Theorem: A quadrilateral is a rectangle if and only if it has four right (congruent) angles.

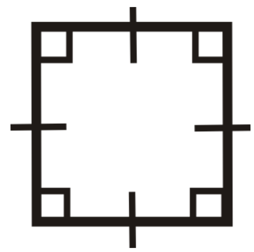


Rhombus Theorem: A quadrilateral is a rhombus if and only if it has four congruent sides.



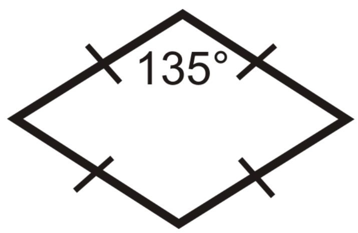
SquareTheorem: A quadrilateral is a square if and only if it has four right angles and four congruent sides.

From the Square Theorem, we can also conclude that a ***square is a rectangle and a rhombus***.

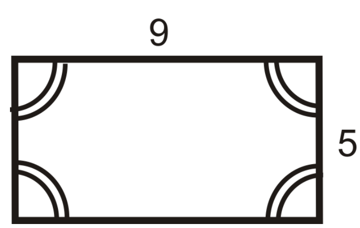


Example 1: What type of parallelogram are the ones below?

a)



b)



Solution:

a) All sides are congruent and one angle is 135°, meaning that the angles are not congruent. By the Rhombus Theorem, this is a rhombus.

b) This quadrilateral has four congruent angles and all the sides are not congruent. By the Rectangle Theorem, this is a rectangle.

Example 2: Is a rhombus SOMETIMES, ALWAYS, or NEVER a square? Explain your reasoning.

Solution: A rhombus has four congruent sides, while a square has four congruent sides and angles. Therefore, a rhombus is only a square when it also has congruent angles. So, a rhombus is SOMETIMES a square.

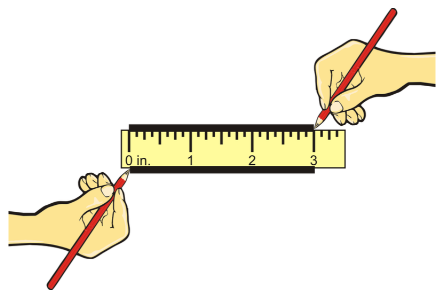
Diagonals in Special Parallelograms

Recall from previous lessons that the ***diagonals in a parallelogram bisect each other***. Therefore, the diagonals of a rectangle, square and rhombus also bisect each other. The diagonals of these parallelograms also have additional properties.

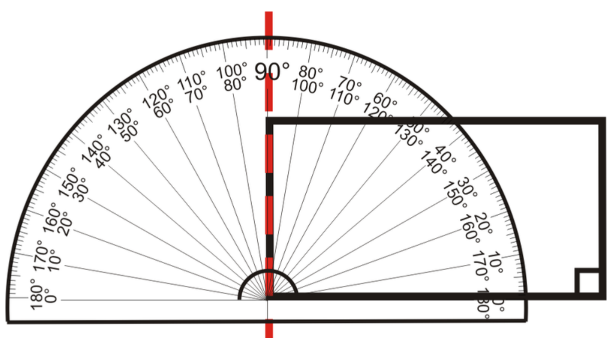
Investigation 6-3: Drawing a Rectangle

Tools Needed: pencil, paper, protractor, ruler

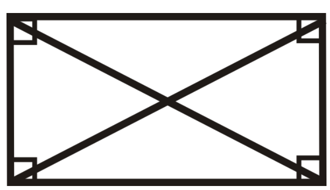
a. Like with Investigation 6-2, draw two lines on either side of your ruler, to ensure they are parallel. Make these lines 3 inches long.



b. Remove the ruler and mark two 90° angles, 2.5 inches apart on the bottom line drawn in Step 1. Then, draw the angles to intersect the top line. This will ensure that all four angles are 90°. Depending on your ruler, the sides should be 2.5 inches and 1 inch.



c. Draw in the diagonals and measure them. What do you discover?



Theorem 6-14: A parallelogram is a rectangle if and only if the diagonals are congruent.

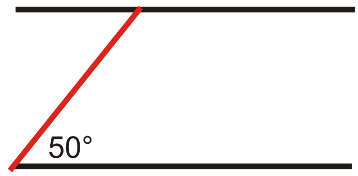
Notice, we did not say *any quadrilateral*. There are quadrilaterals that have congruent diagonals and are not parallelograms.

Investigation 6-4: Drawing a Rhombus

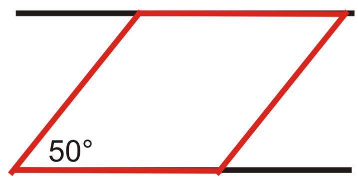
Tools Needed: pencil, paper, protractor, ruler

a. Like with Investigation 6-2 and 6-3, draw two lines on either side of your ruler, to ensure they are parallel. Make these lines 3 inches long.

b. Remove the ruler and mark a 50° angle, at the left end of the bottom line drawn in Step 1. Draw the other side of the angle and make sure it intersects the top line. Measure the length of this side.

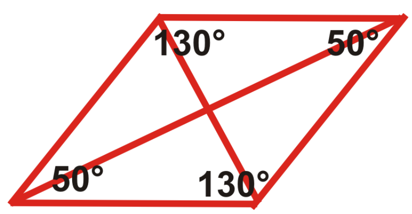


c. The measure of the diagonal (red) side should be about 1.3 inches (if your ruler is 1 inch wide). Mark this length on the bottom line and the top line from the point of intersection with the 50° angle. Draw in the fourth side. It will connect the two endpoints of these lengths.



d. By the way we drew this parallelogram; it is a rhombus because all four sides are 1.3 inches long. Draw in the diagonals.

Measure the angles created by the diagonals: the angles at their point of intersection and the angles created by the sides and each diagonal. You should find the measure of 12 angles total. What do you discover?

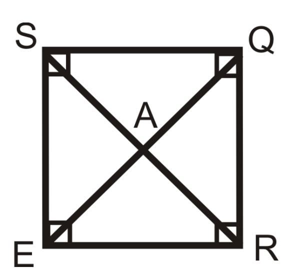


Theorem 6-15: A parallelogram is a rhombus if and only if the diagonals are perpendicular.

Theorem 6-16: A parallelogram is a rhombus if and only if the diagonals bisect each angle.

There are no theorems about the diagonals of a square. We know that a square is a rhombus and a rectangle. So, the diagonals of a square have the properties of a rhombus and a rectangle.

Example 3: List everything you know about the square $SQRE$.



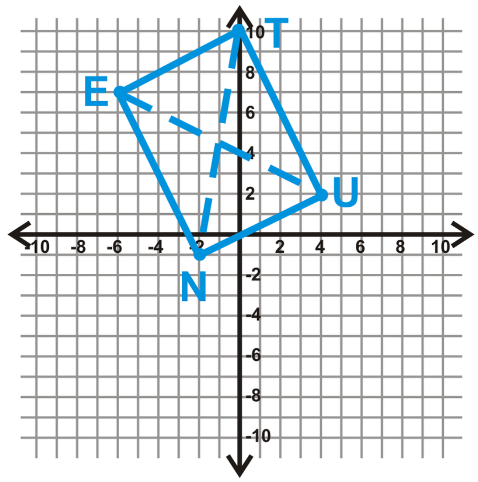
Solution: A square has all the properties of a parallelogram, rectangle and rhombus.

TABLE 6.8:

Properties of Parallelograms	*Properties of Rhombuses*	*Properties of Rectangles*
• $\overline{SQ} \parallel \overline{ER}$	• $\overline{SQ} \cong \overline{ER} \cong \overline{SE} \cong \overline{QR}$	• $\angle SER \cong \angle SQR \cong \angle QSE \cong \angle QRE$
• $\overline{SE} \parallel \overline{QR}$	• $\overline{SR} \perp \overline{QE}$	• $\overline{SR} \cong \overline{QE}$
• $\overline{SQ} \cong \overline{ER}$	• $\angle SEQ \cong \angle QER \cong \angle SQE \cong \angle EQR$	• $\overline{SA} \cong \overline{AR} \cong \overline{QA} \cong \overline{AE}$
• $\overline{SE} \cong \overline{QR}$	• $\angle QSR \cong \angle RSE \cong \angle QRS \cong \angle SRE$	
• $\overline{SA} \cong \overline{AR}$		
• $\overline{QA} \cong \overline{AE}$		
• $\angle SER \cong \angle SQR$		
• $\angle QSE \cong \angle QRE$		

Parallelograms in the Coordinate Plane

Example 4: Determine what type of parallelogram $TUNE$ is: $T(0,10)$, $U(4,2)$, $N(-2,-1)$, and $E(-6,7)$.



Solution: First, let's double-check and make sure the diagonals bisect each other.

Midpoint of $EU = \left(\frac{-6+4}{2}, \frac{7+2}{2}\right) = (-1, 4.5)$

Midpoint of $TN = \left(\frac{0-2}{2}, \frac{10-1}{2}\right) = (-1, 4.5)$

Now, let's see if the diagonals are equal. If they are, then $TUNE$ is a rectangle.

$$\begin{aligned} EU &= \sqrt{(-6-4)^2+(7-2)^2} \\ &= \sqrt{(-10)^2+5^2} \\ &= \sqrt{100+25} \\ &= \sqrt{125} \end{aligned} \qquad \begin{aligned} TN &= \sqrt{(0+2)^2+(10+1)^2} \\ &= \sqrt{2^2+11^2} \\ &= \sqrt{4+121} \\ &= \sqrt{125} \end{aligned}$$

If the diagonals are perpendicular, then $TUNE$ is a square.

Slope of $EU = \frac{7-2}{-6-4} = -\frac{5}{10} = -\frac{1}{2}$ Slope of $TN = \frac{10+1}{0+2} = \frac{11}{2}$

The slope of EU is not the opposite reciprocal of the slope of TN, so we can conclude that $TUNE$ is not a square, it is a rectangle.

Here are the steps to determine if a quadrilateral is a parallelogram, rectangle, rhombus, or square.

1. See if the diagonals bisect each otherby using the midpoint formula.

Yes: ***Parallelogram***, continue to #2. **No:** A ***quadrilateral***, done.

2. Determine if the diagonals are equalby using the distance formula.

Yes: ***Rectangle***, skip to #4. **No:** Could be a rhombus, continue to #3.

3. Determine if the sides are congruentby using the distance formula.

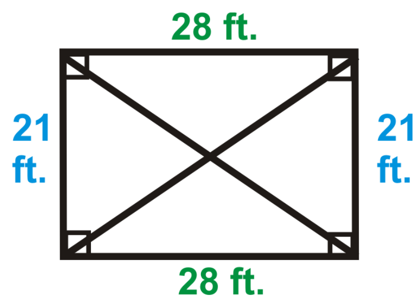
Yes: ***Rhombus***, done. **No:** ***Parallelogram***, done.

4. See if the diagonals are perpendicularby finding their slopes.

Yes: ***Square***, done. **No:** ***Rectangle***, done.

NOTE: This is just one list of steps to take to determine what type of parallelogram a quadrilateral is. There are several other steps that you could take based on the theorems we have learned.

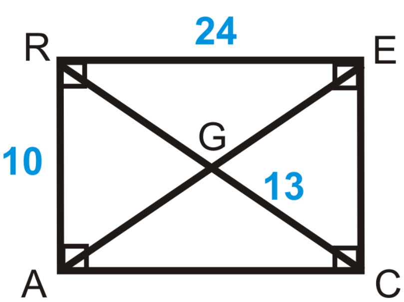
Know What? Revisited In order for the patio to be a rectangle, first the opposite sides must be congruent. So, two sides are 21ft and two are 28 ft. To ensure that the parallelogram is a rectangle *without* measuring the angles, the diagonals must be equal. You can find the length of the diagonals by using the Pythagorean Theorem.



$$d^2 = 21^2 + 28^2 = 441 + 784 = 1225$$
$$d = \sqrt{1225} = 35\ ft$$

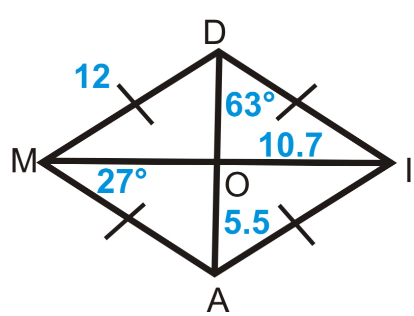
Review Questions

1. *RACE* is a rectangle. Find:



   a. RG
   b. AE
   c. AC
   d. EC
   e. $m\angle RAC$

2. *DIAM* is a rhombus. Find:

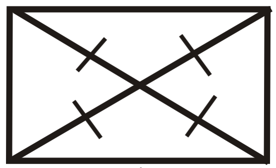


   a. MA
   b. MI
   c. DA
   d. $m\angle DIA$
   e. $m\angle MOA$

3. Draw a square and label it $CUBE$. Mark the point of intersection of the diagonals Y. Find:

a. $m\angle UCE$
b. $m\angle EYB$
c. $m\angle UBY$
d. $m\angle UEB$

For questions 4-12, determine if the quadrilateral is a parallelogram, rectangle, rhombus, square or none. Explain your reasoning.

4.


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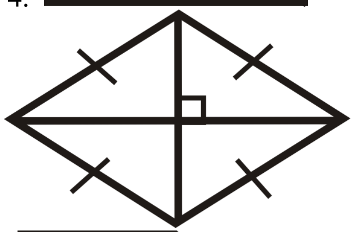

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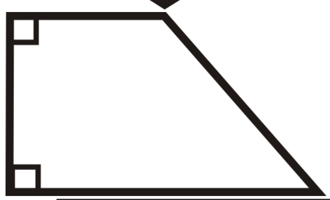

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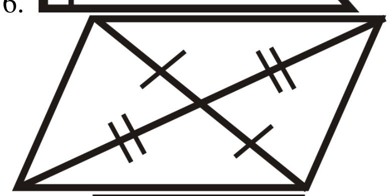

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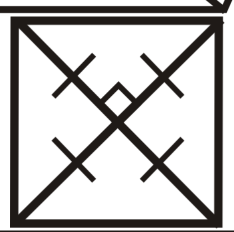

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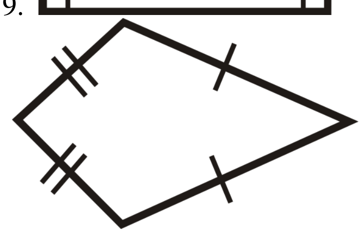
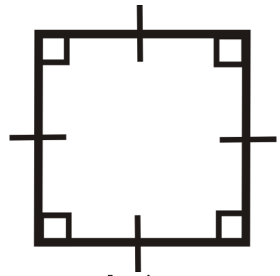
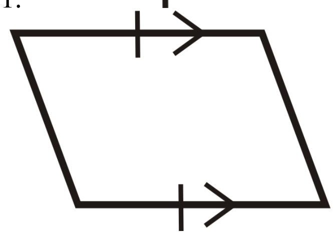


11. 

12. 

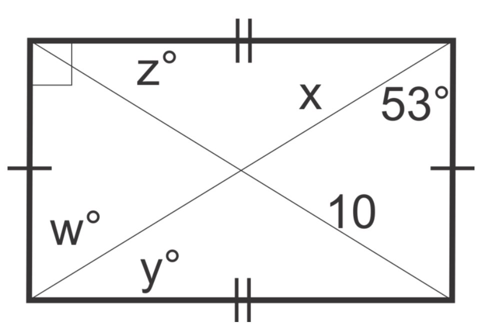
For questions 13-18 determine if the following are ALWAYS, SOMETIME, or NEVER true. Explain your reasoning.

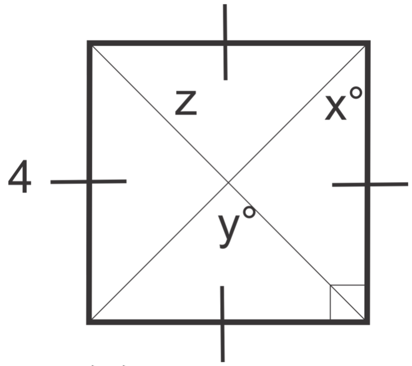
13. A rectangle is a rhombus.
14. A square is a parallelogram.
15. A parallelogram is regular.
16. A square is a rectangle.
17. A rhombus is equiangular.
18. A quadrilateral is a pentagon.

For questions 19-22, determine what type of quadrilateral *ABCD* is.

19. $A(-2,4),\ B(-1,2),\ C(-3,1),\ D(-4,3)$
20. $A(-2,3),\ B(3,4),\ C(2,-1),\ D(-3,-2)$
21. $A(1,-1),\ B(7,1),\ C(8,-2),\ D(2,-4)$
22. $A(10,4),\ B(8,-2),\ C(2,2),\ D(4,8)$
23. ***Writing*** From 19-22, you used the coordinate plane to determine what type of quadrilateral *ABCD* is. Other than the method described in this section, describe another way to determine if *ABCD* is a square.

For problems 24-26, find the value of each variable in the figures.

24. 



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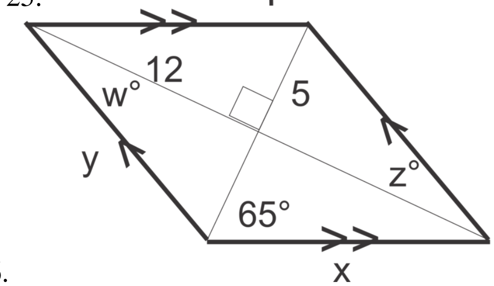


26.

27. Given: $ABCD$ is a rectangle W, X, Y and Z are midpoints of $\overline{BC}$, $\overline{AB}$, $\overline{AD}$ and $\overline{CD}$ respectively Prove: quadrilateral $WXYZ$ is a rhombus

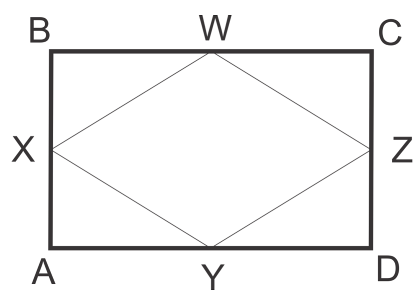


TABLE 6.9:

Statements	***Reasons***
7. $ABCD$ is a rectangle	1. Given
8. $\overline{BW} \cong \overline{WC}$, _____, _____, _____	2. Definition of a midpoint
9. $BD = AC$	3.
10. $\overline{XY}$ is a midsegment in $\triangle ABD$ ________, ________, ________	4. Definition of a midsegment in a triangle
11. $XY = \frac{1}{2}BD = WZ$ and ________	5. Midsegment in a triangle is half the length of the parallel side.
12. $\frac{1}{2}BD = \frac{1}{2}AC$	6.
13. $XY = WZ = YZ = XW$	7.
14. $WXYZ$ is a rhombus	8.

28. Explain why a quadrilateral formed by the segments connecting the midpoints of the sides in a rhombus is always a rectangle.
29. Explain why a quadrilateral formed by the segments connecting the midpoints of the sides in a square is always a square.
30. Construct a rhombus with diagonals of lengths 2 inches and 1.5 inches.
31. Construct a rectangle with diagonal length inches.

Review Queue Answers

a. A rectangle has all equal angles and opposite sides are congruent. It is a parallelogram.
b. A square has equal angles and sides. It is a parallelogram and a rectangle (and a rhombus).
c. Possibilities: picture frame, door, baseball diamond, windows, walls, floor tiles, book cover, pages/paper, table/desk top, black/white board, the diamond suit (in a deck of cards).

6.5 Trapezoids and Kites

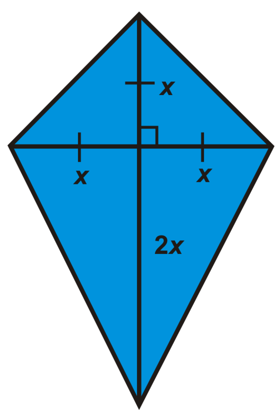
Learning Objectives

- Define and find the properties of trapezoids, isosceles trapezoids, and kites.
- Discover the properties of midsegments of trapezoids.
- Plot trapezoids, isosceles trapezoids, and kites in the coordinate plane.

Review Queue

a. Draw a quadrilateral with one set of parallel lines.
b. Draw a quadrilateral with one set of parallel lines and two right angles.
c. Draw a quadrilateral with one set of parallel lines and two congruent sides.
d. Draw a quadrilateral with one set of parallel lines and three congruent sides.
e. Draw a quadrilateral with two sets of congruent sides and no parallel sides.

Know What? A traditional kite, seen at the right, is made by placing two pieces of wood perpendicular to each other and one piece of wood is bisected by the other. The typical dimensions are included in the picture. If you have two pieces of wood, 36 inches and 54 inches, determine the values of x and $2x$. Then, determine how large a piece of canvas you would need to make the kite (find the perimeter of the kite).



Trapezoids

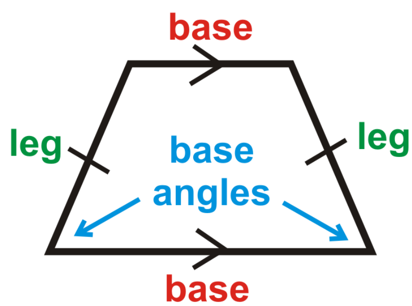
Unlike parallelograms, trapezoids have only one set of parallel lines. The other two sides have no restrictions.

Trapezoid: A quadrilateral with exactly one pair of parallel sides.

Examples look like:

Isosceles Trapezoid: A trapezoid where the non-parallel sides are congruent.

The third trapezoid above is an example of an isosceles trapezoid. Think of it as an isosceles triangle with the top cut off. Isosceles trapezoids also have parts that are labeled much like an isosceles triangle. Both parallel sides are called bases.

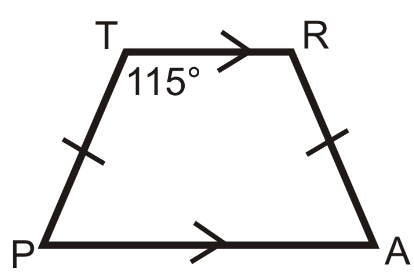


Isosceles Trapezoids

Previously, we introduced the Base Angles Theorem with isosceles triangles. The theorem states that in an isosceles triangle, the two base angles are congruent. This property holds true for isosceles trapezoids. ***The two angles along the same base in an isosceles trapezoid will also be congruent.*** This creates two pairs of congruent angles.

Theorem 6-17: The base angles of an isosceles trapezoid are congruent.

Example 1: Look at trapezoid $TRAP$ below. What is $m\angle A$?

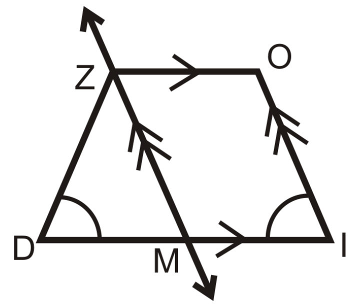


Solution: $TRAP$ is an isosceles trapezoid. So, $m\angle R = 115^\circ$, by Theorem 6-17. To find $m\angle A$, set up an equation.

$$\begin{aligned}
115^\circ + 115^\circ + m\angle A + m\angle P &= 360^\circ \\
230^\circ + 2m\angle A &= 360^\circ \rightarrow m\angle A = m\angle P \\
2m\angle A &= 130^\circ \\
m\angle A &= 65^\circ
\end{aligned}$$

Notice that $m\angle R + m\angle A = 115^\circ + 65^\circ = 180^\circ$. These angles will always be supplementary because of the Consecutive Interior Angles Theorem from Chapter 3. Therefore, the two angles along the same leg (or non-parallel side) are always going to be supplementary. Only in isosceles trapezoids will opposite angles also be supplementary.

Example 2: Write a two-column proof.



Given: Trapezoid $ZOID$ and parallelogram $ZOIM$

$\angle D \cong \angle I$

Prove: $\overline{ZD} \cong \overline{OI}$

Solution:

TABLE 6.10:

Statement	*Reason*
1. Trapezoid $ZOID$ and parallelogram $ZOIM$, $\angle D \cong \angle I$	Given
2. $\overline{ZM} \cong \overline{OI}$	Opposite Sides Theorem
3. $\angle I \cong \angle ZMD$	Corresponding Angles Postulate
4. $\angle D \cong \angle ZMD$	Transitive PoC
5. $\overline{ZM} \cong \overline{ZD}$	Base Angles Converse
6. $\overline{ZD} \cong \overline{OI}$	Transitive PoC

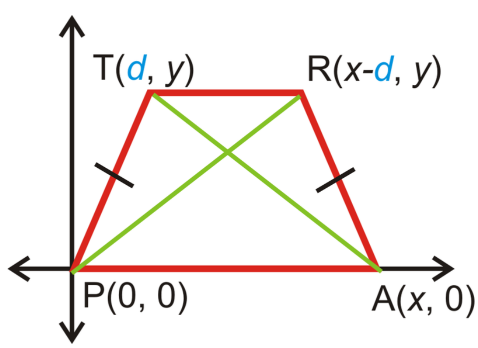
In this example we proved the converse of Theorem 6-17.

Theorem 6-17 Converse: If a trapezoid has congruent base angles, then it is an isosceles trapezoid.

Next, we will investigate the diagonals of an isosceles triangle. Recall, that the diagonals of a rectangle are congruent AND they bisect each other. The diagonals of an isosceles trapezoid are also congruent, but they do NOT bisect each other.

Isosceles Trapezoid Diagonals Theorem: The diagonals of an isosceles trapezoid are congruent.

Example 3: Show $TA = RP$.



Solution: This is an example of a coordinate proof. Here, we will use the distance formula to show that $TA = RP$, but with letters instead of numbers for the coordinates.

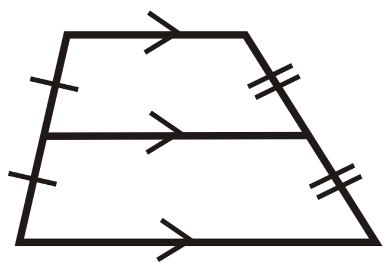
$$TA = \sqrt{(x-d)^2 + (0-y)^2} \qquad RP = \sqrt{(x-d-0)^2 + (y-0)^2}$$
$$= \sqrt{(x-d)^2 + (-y)^2} \qquad = \sqrt{(x-d)^2 + y^2}$$
$$= \sqrt{(x-d)^2 + y^2}$$

Notice that we end up with the same thing for both diagonals. This means that the diagonals are equal and we have proved the theorem.

Midsegment of a Trapezoid

Midsegment (of a trapezoid): A line segment that connects the midpoints of the non-parallel sides.

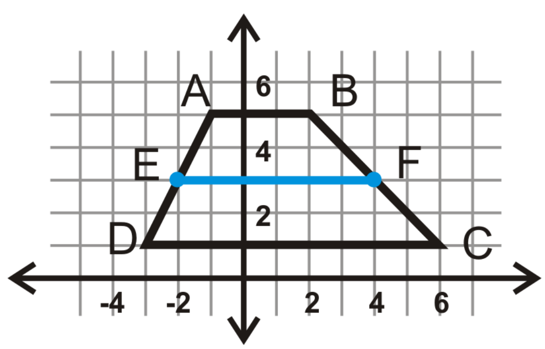
There is only one midsegment in a trapezoid. It will be parallel to the bases because it is located halfway between them. Similar to the midsegment in a triangle, where it is half the length of the side it is parallel to, the midsegment of a trapezoid also has a link to the bases.



Investigation 6-5: Midsegment Property

Tools Needed: graph paper, pencil, ruler

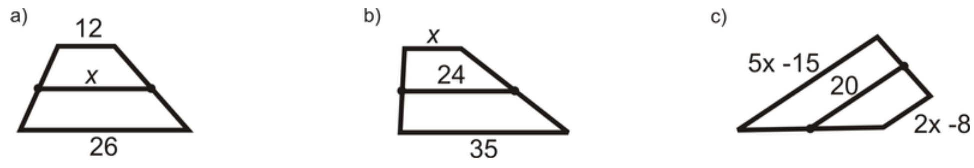
a. Draw a trapezoid on your graph paper with vertices $A(-1,5)$, $B(2,5)$, $C(6,1)$ and $D(-3,1)$. Notice this is NOT an isosceles trapezoid.



b. Find the midpoint of the non-parallel sides either by using slopes or the midpoint formula. Label them E and F. Connect the midpoints to create the midsegment.
c. Find the lengths of AB, EF, and CD. Can you write a formula to find the midsegment?

Midsegment Theorem: The length of the midsegment of a trapezoid is the average of the lengths of the bases, or $EF = \frac{AB+CD}{2}$.

Example 4: ***Algebra Connection*** Find x. All figures are trapezoids with the midsegment.



Solution:

a) x is the average of 12 and 26. $\frac{12+26}{2} = \frac{38}{2} = 19$

b) 24 is the average of x and 35.

$$\begin{aligned} \frac{x+35}{2} &= 24 \\ x+35 &= 48 \\ x &= 13 \end{aligned}$$

c) 20 is the average of $5x-15$ and $2x-8$.

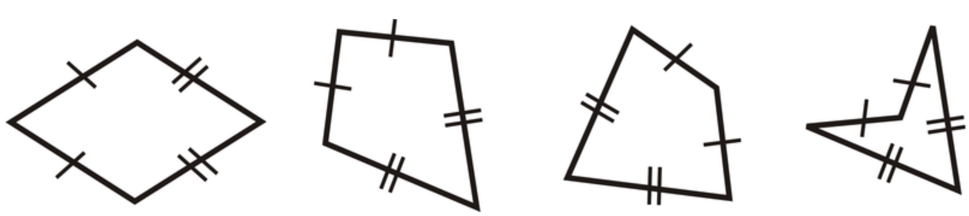
$$\begin{aligned} \frac{5x-15+2x-8}{2} &= 20 \\ 7x-23 &= 40 \\ 7x &= 63 \\ x &= 9 \end{aligned}$$

Kites

The last quadrilateral we will study is a kite. Like you might think, it looks like a traditional kite that is flown in the air.

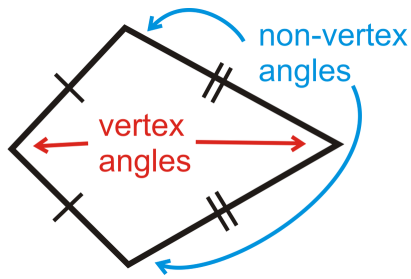
Kite: A quadrilateral with two sets of adjacent congruent sides.

A few examples:



From the definition, a kite is the only quadrilateral that we have discussed that could be concave, as with the case of the last kite. If a kite is concave, it is called a ***dart***.

The angles between the congruent sides are called ***vertex angles***. The otherangles are called ***non-vertex angles***. If we draw the diagonal through the vertex angles, we would have two congruent triangles.



Given: $KITE$ with $\overline{KE} \cong \overline{TE}$ and $\overline{KI} \cong \overline{TI}$

Prove: $\angle K \cong \angle T$

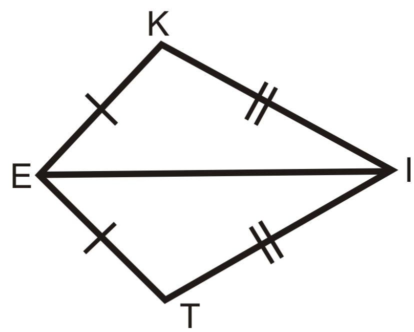


TABLE 6.11:

Statement	***Reason***
1. $\overline{KE} \cong \overline{TE}$ and $\overline{KI} \cong \overline{TI}$	Given
2. $\overline{EI} \cong \overline{EI}$	Reflexive PoC
3. $\triangle EKI \cong \triangle ETI$	SSS
4. $\angle K \cong \angle T$	CPCTC

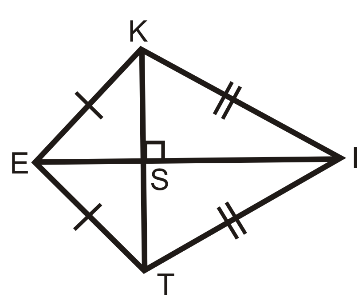
Theorem 6-21: The non-vertex angles of a kite are congruent.

Theorem 6-22: The diagonal through the vertex angles is the angle bisector for both angles.

The proof of Theorem 6-22 is very similar to the proof above for Theorem 6-21. If we draw in the other diagonal in $KITE$ we find that the two diagonals are perpendicular.

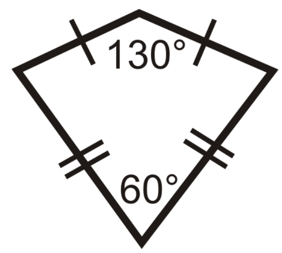
Kite Diagonals Theorem: The diagonals of a kite are perpendicular.

To prove that the diagonals are perpendicular, look at $\triangle KET$ and $\triangle KIT$. Both of these triangles are isosceles triangles, which means $\overline{EI}$ is the perpendicular bisector of $\overline{KT}$ (the Isosceles Triangle Theorem, Chapter 4). Use this information to help you prove the diagonals are perpendicular in the review questions.

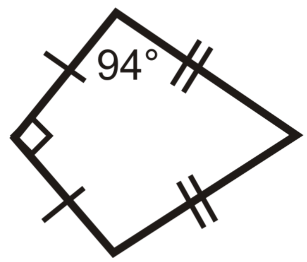


Example 5: Find the other two angle measures in the kites below.

a)



b)



Solution:

a) The two angles left are the non-vertex angles, which are congruent.

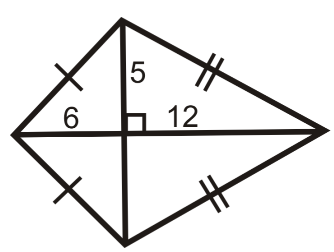
$$\begin{aligned} 130^\circ + 60^\circ + x + x &= 360^\circ \\ 2x &= 170^\circ \\ x &= 85^\circ \qquad \text{Both angles are } 85^\circ. \end{aligned}$$

b) The other non-vertex angle is also 94°. To find the fourth angle, subtract the other three angles from 360°.

$$\begin{aligned} 90^\circ + 94^\circ + 94^\circ + x &= 360^\circ \\ x &= 82^\circ \end{aligned}$$

Be careful with the definition of a kite. The congruent pairs are distinct. This means that ***a rhombus and square cannot be a kite***.

Example 6: Use the Pythagorean Theorem to find the length of the sides of the kite.

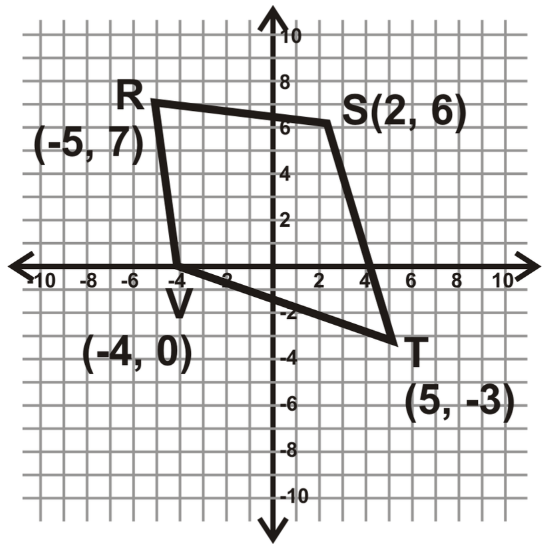


Solution: Recall that the Pythagorean Theorem is $a^2 + b^2 = c^2$, where c is the hypotenuse. In this kite, the sides are all hypotenuses.

$$\begin{aligned} 6^2+5^2&=h^2 \\ 36+25&=h^2 \\ 61&=h^2 \\ \sqrt{61}&=h \end{aligned} \qquad \begin{aligned} 12^2+5^2&=j^2 \\ 144+25&=j^2 \\ 169&=j^2 \\ 13&=j \end{aligned}$$

Kites and Trapezoids in the Coordinate Plane

Example 7: Determine what type of quadrilateral $RSTV$ is. Simplify all radicals.



Solution: There are two directions you could take here. First, you could determine if the diagonals bisect each other. If they do, then it is a parallelogram and you could proceed like the previous section. Or, you could find the lengths of all the sides. Let's do this option.

$$\begin{aligned} RS &= \sqrt{(-5-2)^2+(7-6)^2} \\ &= \sqrt{(-7)^2+1^2} \\ &= \sqrt{50} = 5\sqrt{2} \end{aligned} \qquad \begin{aligned} ST &= \sqrt{(2-5)^2+(6-(-3))^2} \\ &= \sqrt{(-3)^2+9^2} \\ &= \sqrt{90} = 3\sqrt{10} \end{aligned}$$

$$\begin{aligned} RV &= \sqrt{(-5-(-4))^2+(7-0)^2} \\ &= \sqrt{(-1)^2+7^2} \\ &= \sqrt{50} = 5\sqrt{2} \end{aligned} \qquad \begin{aligned} VT &= \sqrt{(-4-5)^2+(0-(-3))^2} \\ &= \sqrt{(-9)^2+3^2} \\ &= \sqrt{90} = 3\sqrt{10} \end{aligned}$$

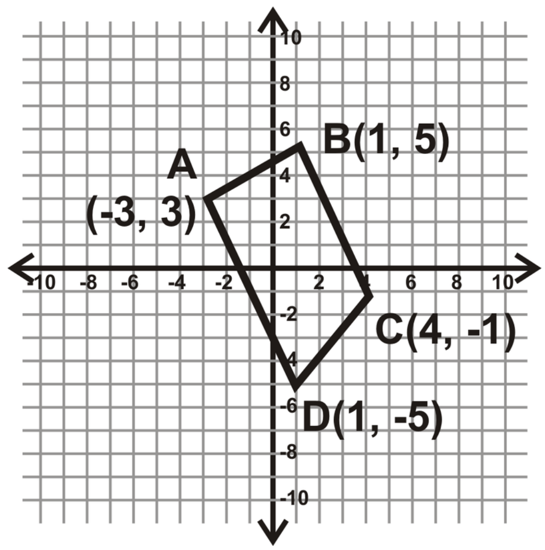
From this we see that the adjacent sides are congruent. Therefore, $RSTV$ is a kite.

Algebra Review: From now on, this text will ask you to "simplify the radical." From Algebra, this means that you pull all square numbers (1, 4, 9, 16, 25, ...) out of the radical. Above $\sqrt{50} = \sqrt{25 \cdot 2}$. We know $\sqrt{25} = 5$, so $\sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}$.

Hint: If you are only given a set of points when determining what type of quadrilateral a figure is, always plot the points and graph. The visual will help you decide which direction to go.

Example 8: Determine what type of quadrilateral $ABCD$ is. $A(-3,3),\ B(1,5),\ C(4,-1),\ D(1,-5)$. Simplify all radicals.

Solution: First, graph $ABCD$. This will make it easier to figure out what type of quadrilateral it is. From the graph, we can tell this is not a parallelogram. Find the slopes of $\overline{BC}$ and $\overline{AD}$ to see if they are parallel.



Slope of $\overline{BC} = \frac{5-(-1)}{1-4} = \frac{6}{-3} = -2$

Slope of $\overline{AD} = \frac{3-(-5)}{-3-1} = \frac{8}{-4} = -2$

We now know $\overline{BC} \parallel \overline{AD}$. This is a trapezoid. To determine if it is an isosceles trapezoid, find AB and CD.

$$\begin{aligned} AB &= \sqrt{(-3-1)^2+(3-5)^2} \\ &= \sqrt{(-4)^2+(-2)^2} \\ &= \sqrt{20} = 2\sqrt{5} \end{aligned} \qquad \begin{aligned} ST &= \sqrt{(4-1)^2+(-1-(-5))^2} \\ &= \sqrt{3^2+4^2} \\ &= \sqrt{25} = 5 \end{aligned}$$

$AB \neq CD$, therefore this is only a trapezoid.

Example 9: Determine what type of quadrilateral $EFGH$ is.

$$E(5,-1),\ F(11,-3),\ G(5,-5),\ H(-1,-3)$$

Solution: To contrast with Example 8, we will not graph this example. Let's find the length of all four sides.

$$\begin{aligned} EF &= \sqrt{(5-11)^2+(-1-(-3))^2} \\ &= \sqrt{(-6)^2+2^2} \\ &= \sqrt{40} = 2\sqrt{10} \end{aligned} \qquad \begin{aligned} FG &= \sqrt{(11-5)^2+(-3-(-5))^2} \\ &= \sqrt{6^2+2^2} \\ &= \sqrt{40} = 2\sqrt{10} \end{aligned}$$

$$GH = \sqrt{(5-(-1))^2+(-5-(-3))^2} \qquad HE = \sqrt{(-1-5)^2+(-3-(-1))^2}$$
$$= \sqrt{6^2+(-2)^2} \qquad = \sqrt{(-6)^2+(-2)^2}$$
$$= \sqrt{40} = 2\sqrt{10} \qquad = \sqrt{40} = 2\sqrt{10}$$

All four sides are equal. That means, this quadrilateral is either a rhombus or a square. The difference between the two is that a square has four 90° angles and congruent diagonals. Let's find the length of the diagonals.

$$EG = \sqrt{(5-5)^2+(-1-(-5))^2} \qquad FH = \sqrt{(11-(-1))^2+(-3-(-3))^2}$$
$$= \sqrt{0^2+4^2} \qquad = \sqrt{12^2+0^2}$$
$$= \sqrt{16} = 4 \qquad = \sqrt{144} = 12$$

The diagonals are not congruent, so $EFGH$ is a rhombus.

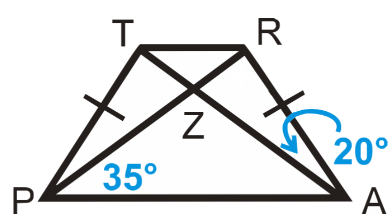
Know What? Revisited If the diagonals (pieces of wood) are 36 inches and 54 inches, x is half of 36, or 18 inches. Then, $2x$ is 36. To determine how large a piece of canvas to get, find the length of each side of the kite using the Pythagorean Theorem.

$$18^2+18^2 = s^2 \qquad 18^2+36^2 = t^2$$
$$324 = s^2 \qquad 1620 = t^2$$
$$18\sqrt{2} \approx 25.5 \approx s \qquad 18\sqrt{5} \approx 40.25 \approx t$$

The perimeter of the kite would be $25.5+25.5+40.25+40.25 = 131.5$ inches or 11 ft, 10.5 in.

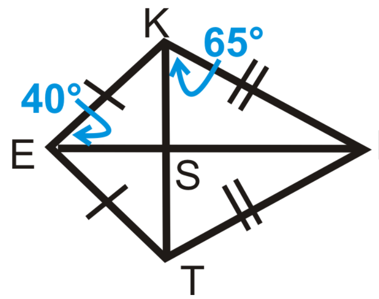
Review Questions

1. $TRAP$ an isosceles trapezoid. Find:



   a. $m\angle TPA$
   b. $m\angle PTR$
   c. $m\angle ZRA$
   d. $m\angle PZA$

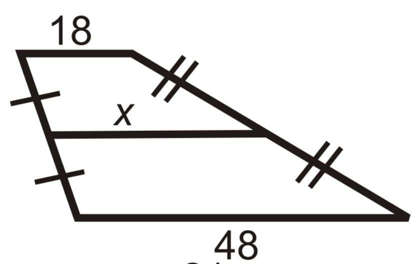
2. $KITE$ is a kite. Find:

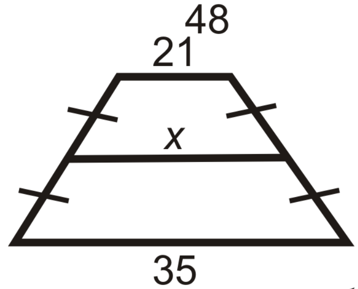


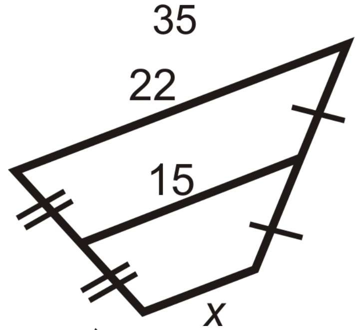
a. $m\angle ETS$
b. $m\angle KIT$
c. $m\angle IST$
d. $m\angle SIT$
e. $m\angle ETI$

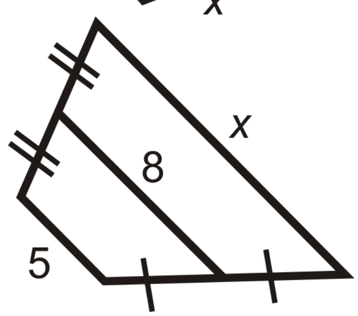
3. ***Writing*** Can the parallel sides of a trapezoid be congruent? Why or why not?
4. ***Writing*** Besides a kite and a rhombus, can you find another quadrilateral with perpendicular diagonals? Explain and draw a picture.
5. ***Writing*** Describe how you would draw or construct a kite.

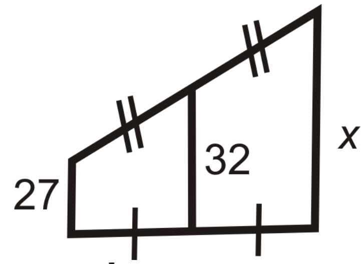
For questions 6-11, find the length of the midsegment or missing side.

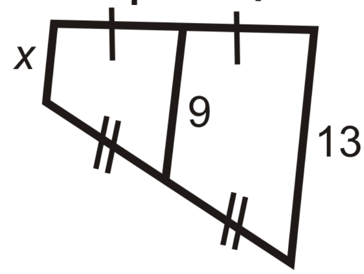
6. 

7. 

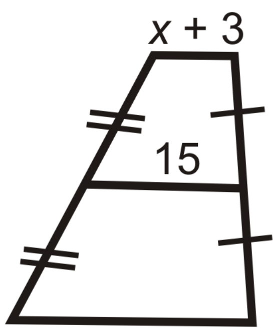
8. 

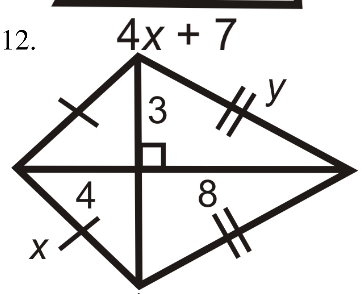
9. 

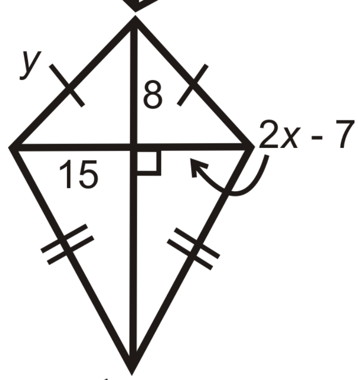
10. 

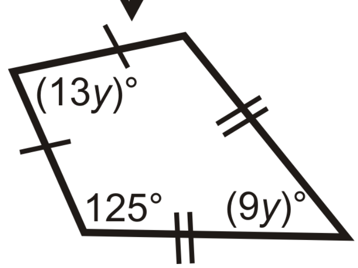
11. 

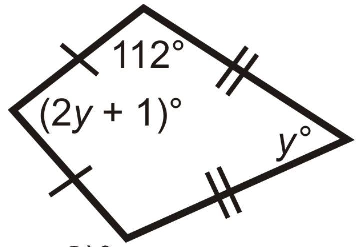
Algebra Connection For questions 12-17, find the value of the missing variable(s). Simplify all radicals.

12. 

13. 

14. 

15. 



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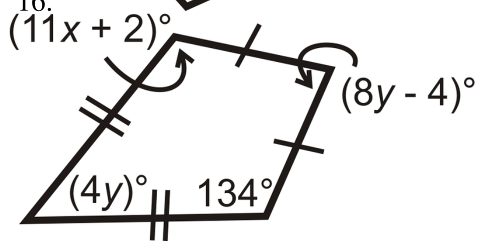
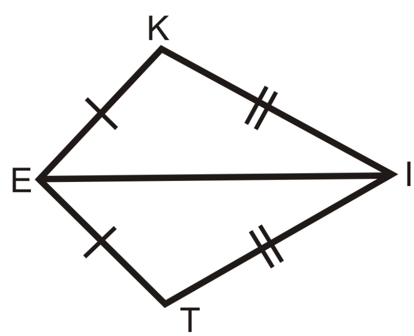


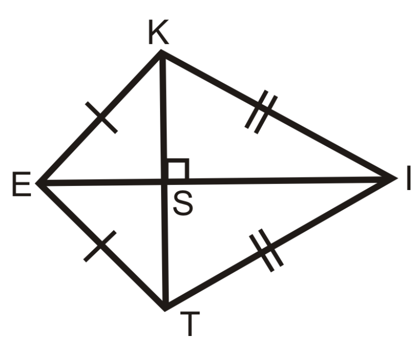
17.

For questions 18-21, determine what type of quadrilateral *ABCD* is. *ABCD* could be any quadrilateral that we have learned in this chapter. If it is none of these, write *none*.

18. $A(1,-2)$, $B(7,-5)$, $C(4,-8)$, $D(-2,-5)$
19. $A(6,6)$, $B(10,8)$, $C(12,4)$, $D(8,2)$
20. $A(-1,8)$, $B(1,4)$, $C(-5,-4)$, $D(-5,6)$
21. $A(5,-1)$, $B(9,-4)$, $C(6,-10)$, $D(3,-5)$
22. $A(-2,2)$, $B(0,1)$, $C(2,2)$, $D(1,5)$
23. $A(-7,4)$, $B(-4,4)$, $C(0,0)$, $D(0,-3)$
24. $A(3,3)$, $B(5,-1)$, $C(7,0)$, $D(5,4)$
25. $A(-4,4)$, $B(-1,2)$, $C(2,4)$, $D(-1,6)$
26. Write a two-column proof of Theorem 6-22. Given: $\overline{KE} \cong \overline{TE}$ and $\overline{KI} \cong \overline{TI}$Prove: $\overline{EI}$ is the angle bisector of $\angle KET$ and $\angle KIT$

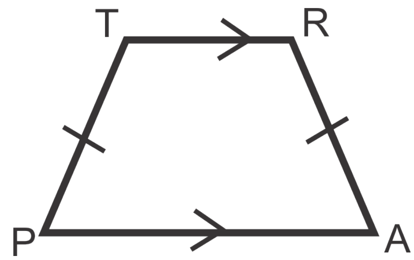


27. Write a two-column proof of the Kite Diagonal Theorem. Given: $\overline{EK} \cong \overline{ET}$, $\overline{KI} \cong \overline{IT}$Prove: $\overline{KT} \perp \overline{EI}$



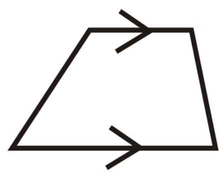
$\star$ Use the hint given earlier in this section.

28. Write a two-column proof of the Isosceles Trapezoid Diagonals Theorem using congruent triangles. Given: $TRAP$ is an isosceles trapezoid with $\overline{TR} \parallel \overline{AP}$. Prove: $\overline{TA} \cong \overline{RP}$

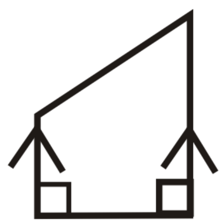


29. Explain why the segments connecting the midpoints of the consecutive sides in a kite will always form a rectangle.
30. Explain why the segments connecting the midpoints of the consecutive sides in an isosceles trapezoid will always form a rhombus.

Review Queue Answers



a.



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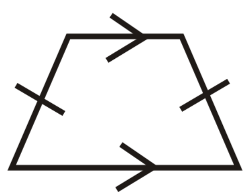
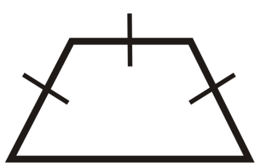


c.



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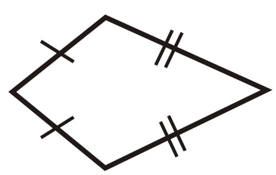


e.

6.6 Chapter 6 Review

Keywords and Theorems

Polygon Sum Formula

For any $n-$gon, the sum of the interior angles is $(n-2)\times 180^\circ$.

Equiangular Polygon Formula

For any equiangular $n-$gon, the measure of each angle is $\frac{(n-2)\times 180^\circ}{n}$.

Regular Polygon

When a polygon is equilateral and equiangular.

Exterior Angle Sum Theorem

The sum of the exterior angles of any polygon is 360°.

Parallelogram

A quadrilateral with two pairs of parallel sides.

Opposite Sides Theorem

If a quadrilateral is a parallelogram, then the opposite sides are congruent.

Opposite Angles Theorem

If a quadrilateral is a parallelogram, then the opposite angles are congruent.

Consecutive Angles Theorem

If a quadrilateral is a parallelogram, then the consecutive angles are supplementary.

Parallelogram Diagonals Theorem

If a quadrilateral is a parallelogram, then the diagonals bisect each other.

Opposite Sides Theorem Converse

If the opposite sides of a quadrilateral are congruent, then the figure is a parallelogram.

Opposite Angles Theorem Converse

If the opposite angles of a quadrilateral are congruent, then the figure is a parallelogram.

Consecutive Angles Theorem Converse

If the consecutive angles of a quadrilateral are supplementary, then the figure is a parallelogram.

Parallelogram Diagonals Theorem Converse

If the diagonals of a quadrilateral bisect each other, then the figure is a parallelogram.

Theorem 6-10

Rectangle Theorem
A quadrilateral is a rectangle if and only if it has four right (congruent) angles.

Rhombus Theorem
A quadrilateral is a rhombus if and only if it has four congruent sides.

Square Theorem
A quadrilateral is a square if and only if it has four right angles and four congruent sides.

Theorem 6-14
A parallelogram is a rectangle if and only if the diagonals are congruent.

Theorem 6-15
A parallelogram is a rhombus if and only if the diagonals are perpendicular.

Theorem 6-16
A parallelogram is a rhombus if and only if the diagonals bisect each angle.

Trapezoid
A quadrilateral with exactly one pair of parallel sides.

Isosceles Trapezoid
A trapezoid where the non-parallel sides are congruent.

Theorem 6-17
The base angles of an isosceles trapezoid are congruent.

Theorem 6-17 Converse
If a trapezoid has congruent base angles, then it is an isosceles trapezoid.

Isosceles Trapezoid Diagonals Theorem
The diagonals of an isosceles trapezoid are congruent.

Midsegment (of a trapezoid)
A line segment that connects the midpoints of the non-parallel sides.

Midsegment Theorem
The length of the midsegment of a trapezoid is the average of the lengths of the bases

Kite
A quadrilateral with two sets of adjacent congruent sides.

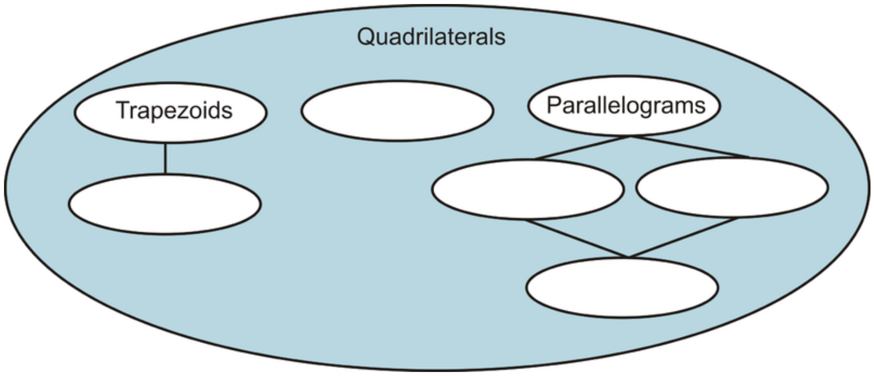
Theorem 6-21
The non-vertex angles of a kite are congruent.

Theorem 6-22
The diagonal through the vertex angles is the angle bisector for both angles.

Kite Diagonals Theorem
The diagonals of a kite are perpendicular.

Quadrilateral Flow Chart

Fill in the flow chart according to what you know about the quadrilaterals we have learned in this chapter.



Sometimes, Always, Never

Determine if the following statements are sometimes, always or never true.

a. A trapezoid is a kite.
b. A square is a parallelogram.
c. An isosceles trapezoid is a quadrilateral.
d. A rhombus is a square.
e. A parallelogram is a square.
f. A square is a kite.
g. A square is a rectangle.
h. A quadrilateral is a rhombus.

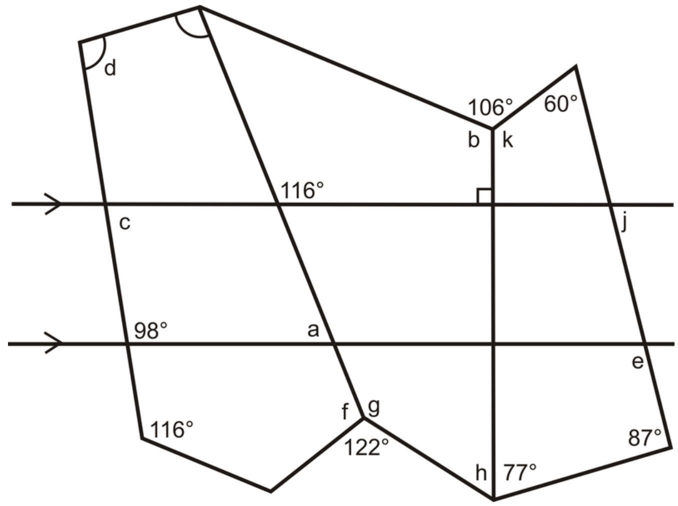
Table Summary

Determine if each quadrilateral has the given properties. If so, write yes or state how many sides (or angles) are congruent, parallel, or perpendicular.

TABLE 6.12:

	Opposite sides $\parallel$	**Diagonals bisect each other**	**Diagonals** $\perp$	**Opposite sides** $\cong$	**Opposite angles** $\cong$	**Consecutive Angles add up to** 180°
Trapezoid						
Isosceles Trapezoid						
Kite						
Parallelogram						
Rectangle						
Rhombus						
Square						

Find the measure of all the lettered angles below. The bottom angle in the pentagon (at the bottom of the drawing) is 138°.



Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9691 .